Sieve analysis and construction: Theory and implementation

Dimitris Exarchos
Centre for Contemporary Music Cultures, Department of Music, Goldsmiths, University of London, UK
d.exarchos@gold.ac.uk

Daniel Jones
Department of Computing, Goldsmiths, University of London, UK
d.jones@gold.ac.uk - http://www.erase.net

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Sieve Theory was used in order to construct symmetries at a desired degree of complexity. This was achieved by the combination of two or more modules, where each module is notated as an ordered pair (*M*, *I*) that indicates a *modulus* (period) and a *residue* (an integer between zero and *M*-1) within that modulus. The abstract image of a sieve is that of a selection of points on a straight line; according to Xenakis "Every well-ordered set can be represented as points on a line, if it is given a reference point for the origin and a length *u* for the unit distance, and this is a sieve" (Xenakis 1992, 268). Modules are combined by the set-theoretical operations of *union*, *intersection* and *complementation*. Given the possibility of multiple notations of the same sieve, the use of each logical operation depends on both the type of formula we choose and on the type of sieve in question. This paper is based on typology of both the different types of sieves as well as the available formulae types for a given sieve. We categorise sieves according to their *symmetry* and *periodicity*. Xenakis refers to these two notions as two distinct levels of identity: in the opening of his article on sieves he talks about "spatial identities" and "identities in time", correspondingly; he then refers to these levels as being internal and external to the sieve (Xenakis 1992, 268). Symmetry is evident in the sieve's intervallic structure and periodicity in its periodic nature.

Sieve Theory is one of the dominant number-theoretical approaches taken by Xenakis to composition, used as a method of constructing sets of points along the integer line. The applications of these sets are manifold: as scales in pitch space, repeating with non-octave periodicities; as a way of structuring sound events in time, to generate novel rhythms and timbres; and, indeed, in any other context that integer collections can be applied.

This paper provides a comprehensive overview of Xenakian Sieve Theory. We begin from its mathematical basis and genesis in Xenakis' thought, distinguishing between types of sieve, formula, and operations that can be carried out on each. We discuss a number of ways in which sieves can be transformed, allowing the derivation of new structures.

We proceed to describe *sieve.maxpat,* an interactive instrument for the exploration of Xenakian Sieve Theory, developed within Max/MSP on the basis of prior work by Ariza (2005). This instrument is intended as an accessible route for musicians to harness sieves within their own work.

Context

Xenakis developed Sieve Theory during his stay in Berlin, having received a Ford Foundation grant to live and work in West Germany in 1963. The theory concerns mainly the creation of scales, arrived at through the combination of residue class sets. The Sieve of Eratosthenes is known as the earliest sieve in mathematics and its technique was fundamental to Xenakis; it had provided a method for filtering elements. The abstract image of the result, the sieve, is that of a series of points on a (topologically) straight line.

The Sieve of Eratosthenes is a method for determining the prime numbers up to a given integer n. Starting with 2 we erase all its multiples; subsequently, erase all the multiples of the next number that has not been erased (3), and so on. We proceed until we reach prime number p, where $p \leqslant \sqrt{n}$. The remaining integers are the prime numbers between 2 and n.

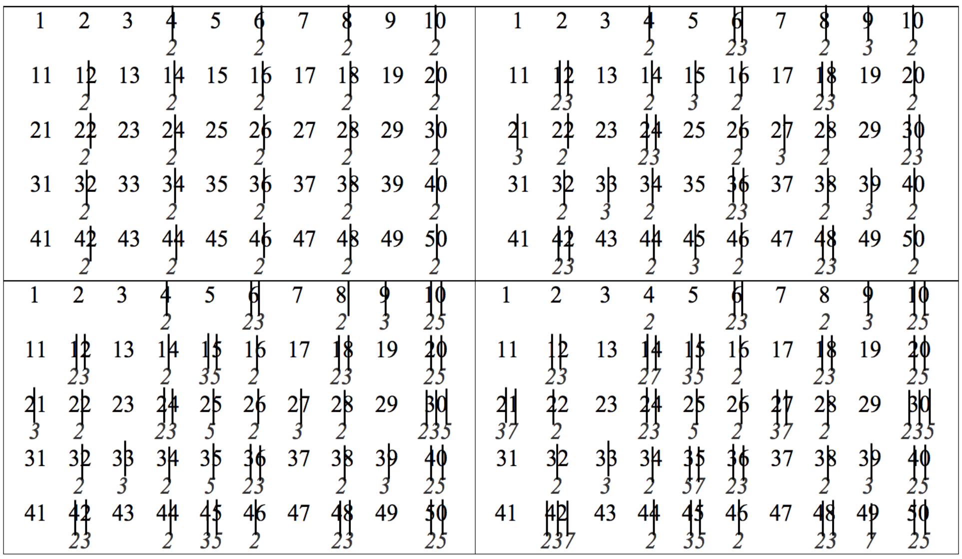

Figure 1. The Sieve of Eratosthenes.

In Figure 1, n = 50. The table consists of four parts (each for one stage of the process) and shows the cross-outs for each element: in the top left part of the table we have erased all the multiples of 2 (every second number), in the top right part all the multiples of 3 (every third number), and so on for 5 and finally 7, which is the greatest prime $p \leqslant \sqrt{50}$.[1]

Xenakis's definition of sieves is general, but clear: "Every well-ordered set can be represented as points on a line, if it is given a reference point for the origin and a length *u* for the unit distance, and this is a sieve" (Xenakis 1992, 268). The theory was used in order to construct symmetries at a desired degree of complexity. This was achieved by the combination of two or more *modules*. A module is notated by an ordered pair (*M*, *I*) that indicates a *modulus* (period) and a *residue* (an integer between zero and *M*-1) within that modulus.

For example, for *M* = 3 and *I* = 1 we have the following module:

$$(3, 1) = \{1\ 4\ 7\ 10\ 13\ \ldots\}.$$

Elements that lie in distance equal to the value of the modulus are said to be *congruent modulo M* or to belong to the same congruence class. In the example, elements 4, 7, and 10 are congruent modulo 3:

$$4 \equiv 7 mod3$$

$$7 \equiv 10 mod3.$$

Logical operations

By applying the set-theoretical operations of union (+), intersection (·), and complementation (–), or a combination of them, one can construct more complex sieves.

The **union** (+) of two modules includes elements that belong to either modules ("either/or").[2] For example

$$(3, 0) + (4, 0) = \{0\ 3\ 4\ 6\ 8\ 9\ 12\ 15\ 16\ 18\ 20\ 21\ 24\ldots\}.$$

The **period** of this sieve is equal to the Lowest Common Multiple (LCM) of 3 and 4, that is, 12.

$$(3, 0) + (4, 0) = \{0\ 3\ 4\ 6\ 8\ 9\}$$

The **intervallic succession**[3] of a sieve is a listing of all its successive intervals:

$$3\ 1\ 2\ 2\ 1\ 3\ 3\ 1\ 2\ 2\ 1\ 3.$$

Intersection (·) includes only the coincidences, elements from both modules ("both/and"). In the case of the major diatonic scale we can choose to represent the sieve either using its period (12 semitones) or using moduli 3 and 4; the operation of union is used in the former case, and a combination of union and intersection in the latter.

$$(12, 0) + (12, 2) + (12, 4) + (12, 5) + (12, 7) + (12, 9) + (12, 11) =$$

$$(4, 0)\cdot(3, 0) + (4, 2)\cdot(3, 2) + (4, 0)\cdot(3, 1) + (4, 1)\cdot(3, 2) + (4, 3)\cdot(3, 1) + (4, 1)\cdot(3, 0) + (4, 3)\cdot(3, 2)$$

Each intersection in the latter formula corresponds to a module in the former. Thus,

$$(12, 0) = (4, 0)\cdot(3, 0)$$

$$(12, 2) = (4, 2)\cdot(3,2)$$

and so on. Within the scope of a single occurrence of a period, an intersection corresponds to a unique point. 3 and 4 are coprime and therefore their product equals the period of 12 semitones. We can now choose to regroup these elements around the modulus of 4 (distributive property):

$$(4, 0) \cdot [(3, 0) + (3, 1)] + (4, 1) \cdot [(3, 0) + (3, 2)] + (4, 3) \cdot [(3, 1) + (3, 2)] + (4, 2)\cdot(3, 2).$$

This alternative formula for the same sieve is aimed at facilitating comparison with other sieves that might share modulus 4, or maybe with other versions of the same sieve.

Complementation (–) includes all the elements that are not members of the original module ("negation"), and is useful to simplify the notation.[4]

For example, $(3,0)+(3,1)=\overline{(3,2)}$.

Types of sieves

There are four types of sieves, two for each of the two criteria of *symmetry* and *periodicity*:

a) Symmetry refers to the intervallic succession of the sieve, which can be either palindromic (symmetric) or not (asymmetric).

b) Periodicity refers to the period of the sieve: this can be either a prime or a composite number.

Symmetric sieves may be expressed merely as a union of two different modules (asymmetric sieves may not). We also call symmetric the sieves that have a palindromic intervallic succession under cyclic transposition, which is essentially a shift of the intervallic succession to the right or to the left.[5] In this sense, all white-key modes 'constitute a unique sieve' (Xenakis 1992: 268) and, since the mode on D is symmetric (with intervallic structure 2 1 2 2 2 1 2), the major diatonic scale is symmetric too.

Sieves with a non-palindromic intervallic succession are called **asymmetric**. The theoretical expression of an asymmetric sieve is impossible without using intersection.

However, decomposing the period is not possible for all sieves. Factorial decomposability refers to intersections of modules and depends on whether the period of the sieve is a prime or a composite number. It follows that sieves whose period is prime cannot be decomposed into factors. Of course, a prime period can be found in any sieve (either symmetric or asymmetric). A prime asymmetric sieve is the one used in *Jonchaies* (1977, for orchestra).[6]

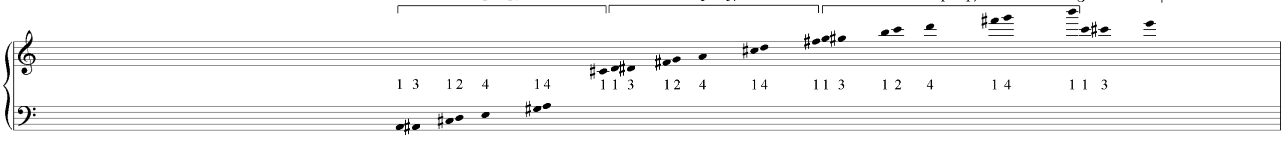

Figure 2. The sieve of *Jonchaies*

Its intervallic succession is asymmetric (it is non-palindromic in any of its cyclic transpositions) and its period is one octave and a perfect 4th, i.e. 17 semitones, which is a prime. This type of sieves can be notated according to their period only, which is equivalent to writing down all its "pitch-classes": {0 1 4 5 7 11 12 16}. In other words, the sieve of *Jonchaies* can be written only as a union of eight modules (one for each element in the range of the period) that share modulus 17 (where 0 = A2):

$$(17, 0) + (17, 1) + (17, 4) + (17, 5) + (17, 7) + (17, 11) + (17, 12) + (17, 16).$$

Xenakis's intention to arrive at a "more hidden" symmetry must therefore refer to **composite asymmetric** sieves (see Xenakis 1992, 269-70). An example of such a sieve is the harmonic minor. But Xenakis's general aesthetic led him to much more complex asymmetric sieves with significantly larger periods, such as 60 or 90 semitones.[7]

Having defined internal symmetry as a palindromic intervallic structure and non-symmetry as non-palindromic, this level of "hidden symmetries", curiously perhaps, seems to refer to intermediate stages between the two extreme poles. These two poles are occupied by symmetric sieves on the one side (either prime or composite) and prime asymmetric sieves on the other; the former are too regular to offer any interesting properties and the latter seem to escape the scope of the theory. In a note of Xenakis' article we read that "it is sometimes necessary and possible to decompose" a modulus (Xenakis 1992, 381). "Possibility" here refers to the period's factorial decomposition (if it is a composite number) and "necessity" to the unveiling of a hidden symmetry whenever it exists.

Types of formulae

Given that there is a indefinite number of alternative formulae when starting with a sieve (as a series of point on a topologically straight line), the following typology refers to formula-types that one can derive from a given sieve. There is no interest as to how the given sieve might have been constructed, and the formula-types are classified according to the information each type reveals. Similarly to sieve-types, there are two criteria and, therefore, four types of formulae. According to whether a formula takes into account the overall period of the sieve or its internal, or *inner periodicities*, it can be either *periodic* or *inner-periodic*. When a formula derives from prime factorisation (of the period or of the internal periodicities), it is called *decomposed*. Finally, a formula which does not derive from prime factorisation and which includes only union of modules is called *simplified*. This paper argues that Xenakis' approach to sieve-based composition has evolved in two phases, and that these phases are reflected in his use of two quite different types of sieve formulae: the periodic decomposed during the 60s and the inner-periodic simplified during the 80s.

Periodic decomposed formula

According to Xenakis, period decomposition enables "comparison among different sieves"; thus we can (a) "study their degree of difference" and (b) "define a notion of distance" (Xenakis 1992, 270). The treatment of composite sieves presupposes that a decision is made on which factors to employ in the decomposition of the period. At first it seems that the only restriction is a matter of convenience: any combination of moduli whose LCM equals the period is sufficient, as long as the difference between the residues in an intersection is divisible by the GCD of the moduli (in order to secure the intersection is not empty; see Xenakis 1992, 272). When the moduli are coprime then the difference between the residues can be of any value – and this is useful in order to be able to apply transformational processes without any restrictions. However, this is only a matter of convenience.

The rationale behind the decomposition of a composite modulus is related to Prime Factorisation.[8] Sieve Theory follows this principle in order to render the building blocks of a sieve (scale, rhythm sequence, etc.), by breaking its period down to elementary moduli (i.e. moduli that cannot be further decomposed). According to the Unique Factorisation Theorem, every natural number either is a prime number itself or can be written as a *unique* product of primes. Therefore, any integer *a* greater than 1 can be uniquely written in the following form:

$$a = p_1^{\,k} \cdot p_2^{\,l} \cdot \ldots \cdot p_k^{\,m}$$

where p_1, p_2, …, p_k are primes.

Any scale can be decomposed to its "elementary" form; this is the level of "hidden symmetry" that Xenakis referred to.

It is easily implied from the theorem that since two numbers a, b are coprime then a^m, b^n are coprime as well (with m, n being any integer greater than 0). In the above form, all numbers (p_1, p_2, … , p_k) are primes themselves and therefore any combination of any two or more, forms a set of coprime numbers. Consequently, any combination of any power of these numbers is still a set of coprime numbers. Thus we see that unique factorisation also secures co-primality.

The prime factors of 12 are $2^2 \cdot 3$. When these two factors correspond to moduli, the literal intersection would be $(2, I_1) \cdot (2, I_2) \cdot (3, I_3)$. It is obvious that $(2, I_1) \cdot (2, I_2)$ is not a valid option. If $I_1 = I_2$ the module intersects with itself: $(2, I_1) = (2, I_2)$; and in the case they are different the intersection is empty.[9] We should therefore resolve any exponentials before treating prime factors as the elementary moduli of a period: $12 = 2^2 \cdot 3 = 4 \cdot 3$. Therefore, $(12, I) = (4, I_1) \cdot (3, I_2)$.

The above means that there is one practical limitation in applying prime factorisation. The resolution of all powers prior to the decomposition of the moduli is not possible for all composite moduli. A modulus that cannot be decomposed is either a prime number, or a *prime power* (the power of a single prime). Such a number, i.e. a prime power, is 16: its prime factorisation gives 2^4. Therefore, although modulus 16 is composite, it cannot be decomposed. Therefore, a sieve whose period is a prime power is non-decomposable and therefore belongs to the same category as prime sieves.

The **decomposed formula** is the one that employs only moduli that are primes or prime powers. These are the elementary moduli that derive from the prime factorisation of the sieve's period. Of this type is the formula that uses moduli 4 and 3 to express a sieve whose period is 12. A more complex decomposed formula is that of the sieve of *Nekuïa*.

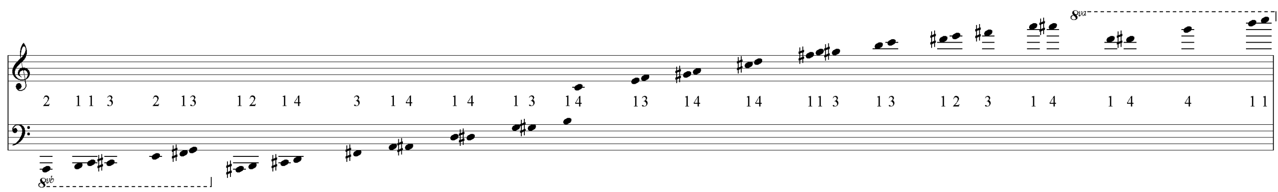

Figure 3. Sieve of *Nekuïa*.

$8_0 \cdot (11_0 + 11_2 + 11_4 + 11_5 + 11_6) + 8_1 \cdot (11_2 + 11_3 + 11_6 + 11_7 + 11_9) + 8_2 \cdot (11_0 + 11_1 + 11_2 + 11_3 + 11_5 + 11_{10}) + 8_3 \cdot (11_1 + 11_2 + 11_3 + 11_4 + 11_{10}) + 8_4 \cdot (11_0 + 11_4 + 11_8) + 8_5 \cdot (11_0 + 11_2 + 11_3 + 11_7 + 11_9 + 11_{10}) + 8_6 \cdot (11_1 + 11_3 + 11_5 + 11_7 + 1_8 + 11_9) + 8_7 \cdot (11_3 + 11_6 + 11_7 + 11_8 + 11_{10})$

The two elementary moduli here are prime power 8 and prime 11 and the period of the sieve is $8 \cdot 11 = 88$ semitones.[10]

Inner periodicities and the simplified formula

In cases where, as with the *Jonchaies* sieve, decomposition is not possible, the periodic formula is necessarily simplified. Similarly, the formula of a sieve might consist only of unions and include moduli that do not derive from the prime factorisation of the period. For example, the following formula for a sieve with period 90, is periodic simplified: (10, 0) + (9, 1).

In order to achieve a *unique* simplified formula (consisting only of unions of modules that do not derive from prime factorisation) we have to look at Xenakis' own algorithm for the generation of the formula, as well as at the notion of inner periodicities.

Although Xenakis's article "Sieves" dates from 1990, the first extended reference to Sieve Theory is found in the final section of "Towards a Metamusic"; an article of 1967, whose unpublished manuscript dates from 1965, titled "Harmoniques (Structures hors-temps)" (see Solomos 2001, 236 & Turner 2005). These two writings reflect two phases of sieve-theoretical and compositional attitude. During the first, from *Akrata* (1964-65) until *Persephassa* (1969), Xenakis used the decomposed formula (which is based on the external period), in order to study its structure and generate transformations. The alternative of a simplified formula appears only in the 1990 article. The progression from the decomposed to the simplified formula, reflects a transition from a certain kind of compositional aesthetics to another.

The internal (intervallic) structure of a sieve had always been more important for Xenakis than its external aspect. Highly irregular and asymmetric intervallic successions, were more important than the length of the period, which in most cases was too long to be audible. Sieve Theory was developed in order to study internal, hidden symmetries. It is important to underline that these symmetries are not found in the intervallic succession as such. Xenakis demonstrated that an elementary modulus is a kind of tempered chromatic scale, with a unit other than the semitone (Xenakis 1992, 195). Furthermore, the elementary moduli are thought both as both symmetries and periodicities (Xenakis 1992, 270). In a decomposed formula these periodicities are shown to coincide (intersection) and join (union) in order to produce the points of the sieve. In a simplified formula the elementary moduli have the form of periodicities that are joined only by union. A combination of elementary modules in a simplified formula can be compared to a combination of several 'chromatic scales' with various units (M) and different starting points (I).

The turning point in Xenakis' sieve-based composition is marked by his orchestral work *Jonchaies* (1977). In the preface to the score he clearly states that the work "deals with pitch 'sieves' (scales) in a new way". As we have shown, the *Jonchaies* sieve is prime asymmetric, which excludes a decomposed formula; and it is a rare case where the period is small enough to be easily audible. It is the first time that such a type is used and this verifies that Xenakis did not rely on the decomposition of the period anymore. In the work, or in Xenakis' language, inside time, the sieve is treated with a particular technique that Solomos has termed "halo sonority" (see Solomos 1996, 84). Xenakis himself described this technique briefly in the preface to the score of *Nekuïa* (1981), as "multiplicities of shifted melodic patterns, like in a kind of artificial reverberation". The result is a kind of heterophony, very different from the set-theoretical treatment of *Herma*'s "symbolic music". Indeed, for Xenakis sieves became timbres rather than pitch sets or scales.

The idea of "multiplicities of shifted patterns" can also be seen in the structure of the sieves following *Jonchaies*. On the surface, these sieves are characterised by an irregular distribution of intervals, between a semitone and a major 3rd, dispersed over the whole range, with rare strings of semitones longer than three. The construction of such sieves is inspired by the Javanese *pelog* with its interlocking fourths; hence the characteristic intervallic succession 1 4 1 semitones, seen already in the *Jonchaies* sieve. Further, in an interview of 1989 Xenakis revealed his interest in choosing intervals that produce 'tension', which for him was a kind of objective category, produced by

> the opposition of large and small intervals – that is, the contrast between something very narrow and something much larger. To maintain this tension along the sieve – in other words in the scale you have chosen – is a tall order. It is also an intriguing problem: none of the parts is to be symmetric – that is, periodic; nor are the ranges to be periodic as compared to the higher or lower ranges, maintaining tension all the while in a different way (Varga 1996, 146).

The contrast of intervals produced by two interlocking fourths is a key example that suggests a move towards the 'simplified' conception of sieves. The intervallic succession of 1 4 1 would be produced by points {0 1 5 6}, which in turn are produced by modules (5, 0) and (5, 1); i.e. two shifted perfect 4hs. Nothing prevents one to add more 'interlocking' modules of various modulus sizes. Each of these modules would thus be equivalent to a different chromatic scale. The result would be a multiplicity of shifted chromatic scales (each with a different unit distance). A simplified formula would be more indicative of such a multiplicity of elementary modules (periodicities), as, unlike the decomposed, it excludes intersection (which in the range of one period produces only a single point). It is exactly the same principle that Xenakis used for the composition of *Jonchaies*. In the following quotation he talks about the rhythmic structures, but the same was applied to the pitch sieves of the following period:

> We can illustrate regular events by points an equal distance apart. On a second, lower parallel line, more points represent other regular patterns with a different time unit, so they are shifted with respect to the first line's points even if they start together. This procedure can be repeated with regular points on other lines. When we hear all these lines together, we obtain a flow of events which consists of a regular intervallic series, but which as a whole is impossible to grasp. Our brain is totally unable to follow such a complicated flow (Xenakis 1996, 148).

The inspiration for *Jonchaies*, the composer comments, comes from the results of his research in sound synthesis for *La légende d'Eer*, of the same year. In the preface to the score of *Jonchaies*, Xenakis writes: "one starts from noise and [...] periodicities are injected to it". Admittedly, this is a possibility offered by stochastics and, in particular, his application of random walks and Brownian movements, which exemplified a reversal of standard sound synthesis. However, the inspiration from electroacoustic to instrumental composition (and vice versa) can be seen, metaphorically, in relation to Sieve Theory as well. The idea of individual periodicities is not extremely different from the original idea of stochastics: individual elements are distributed in such a way that are not intended to be perceived as such, but to create a "multitude of sounds, seen as a totality" (Xenakis 1992, 9).

At the end of "Sieves" Xenakis argues that the inverse, that is the application of Sieve Theory to sound synthesis, is "quite conceivable" (Xenakis 1992, 276). Extending the metaphor of injected periodicities, we could say that the simple modules in a simplified formula represent the individual, internally hidden periodicities (or symmetries, or regularities) of a sieve. In the same sense, it is more than conceivable, instead of starting from noise, to start from the total chromatic throughout the audible range and "inject periodicities" to it in order to construct a sieve that produces a certain timbre. A formula that accounts for these periodicities is not the one whose starting point is the decomposition of the sieve's period. In other words, when a formula ignores the overall period, the elementary moduli account straight for the points of the sieve. Since each module in a simplified formula represents a regularity, then every irregular scale can be analysed as a multiplicity of regularities. This is also the fundamental idea behind the Sieve of Eratosthenes. The sequence of prime numbers has no known pattern; it appears purely random and irregular and the Sieve of Eratosthenes provides a simple method of achieving this irregularity by an algorithmic process that deduces all regular patterns.

Inner-periodic simplified formula

The redundancy of simplified formulae can be overcome by checking every single point of the sieve and assigning to it the smallest possible modulus. More specifically, we can find for each point the module with the smallest modulus, that either starts on this point or produces it later, without producing any "parasitical" points. Xenakis arrived at such a unique simplified formula, by constructing an algorithm, which goes through the following steps:

a) Starting with the lowest point of the sieve (I_n) and with $M = 1$, we test each point for inclusion in module (M, I) and check if:

   i. (M, I) produces only points that belong to the given sieve, and

   ii. (M, I) produces at least one of the not-yet-covered points of the given sieve.

b) If (i) is not satisfied we pass on to $M + 1$. If it is satisfied we keep the module and check if (ii) is satisfied: if yes, we keep the module and pass onto the next point (I_{n+1}); if not, we ignore the module and pass onto I_{n+1}.

c) We stop when each point of the sieve has been covered by a module.

This is the algorithm that Xenakis implemented in his program for the 'Generation of the logical formula of a sieve from a series of points on a line', published in 'Sieves'. Xenakis' own description, according to which 'each point is considered as a point of departure', might be slightly confusing. The algorithm does not merely look for the smallest modulus that *departs* from the point under consideration, but for the smallest modulus that starts at a point *smaller than its own size* and that *produces* the point under consideration (unless this point is located early enough in the sieve that is itself the starting point of the module). Let us take one example: if the point under consideration is 22, the program starts checking, from the bottom up, the modules whose members include 22: (1, 0), (2, 0), (3, 1), (4, 2), (5, 2) and so on, passing on to $M + 1$, until it finds a module that satisfies the following: a) includes point 22) its members all belong to the given sieve and c) it covers at least one of the not-yet-encountered points of the sieve. If no such module is found with $M \leq 22$ (while $M > I$), then it looks for $M > 22$ with $I = 22$. Afterwards, it checks the redundancy of the module; when all points have been covered, it computes the period of the sieve (as the LCM of all moduli) and finally displays the formula.[11]

Therefore, the unique simplified formula that the above algorithm yields, represents the inner periodicities of the sieve, in the form of modules with the smallest possible modulus that account for the given sieve point by point, from the bottom up. We call this, an *inner-periodic* simplified formula. The inner-periodic simplified formula for the sieve of *Nekuïa* is

$24_0 + 14_2 + 22_3 + 31_4 + 28_7 + 29_9 + 19_{10} + 25_{13} + 24_{14} + 26_{17} + 23_{21} + 24_{10} + 30_9 + 35_{17} + 29_{24} + 32_{25} + 30_{29} + 26_{21} + 30_{17} + 31_{16}$.

The information that such a formula type reveals relates to Xenakis' original suggestion of a degree of symmetry. Given the range *n* and an average density *D* of a sieve (number of points / *n*), the larger the number of modules in a formula, the more asymmetric the sieve. The above formula shows clearly that the *Nekuïa* sieve (in fact all Xenakis' sieves of the second phase) is much more asymmetric than, say, a sieve of the same *n* and *D* with two modules, or with one module (i.e. a whole-tone scale that extends for 88 semitones).[12]

Inner-periodic decomposed formula

If we decompose the composite moduli above, we get the inner-periodic decomposed formula:[13]

$8_0 \cdot 3_0 + 2_0 \cdot 7_2 + 2_1 \cdot 11_3 + 31_4 + 4_3 \cdot 7_0 + 29_9 + 19_{10} + 25_{13} + 8_6 \cdot 3_2 + 2_1 \cdot 13_4 + 23_{21} + 8_2 \cdot 3_1 + 2_1 \cdot 3_0 \cdot 5_4 + 5_2 \cdot 7_3 + 29_{24} + 32_{25} + 2_1 \cdot 3_2 \cdot 5_4 + 2_1 \cdot 13_8 + 2_1 \cdot 3_2 \cdot 5_2 + 31_{16}$.

It is obvious from the above, that this type of formula presupposes that the inner-periodic simplified formula has been appropriately found beforehand, through Xenakis' algorithm.

Metabolae (transformations) of sieves

One of the aims that underlie the theoretical expression of a sieve (formula) is to enable transformations or, to use Xenakis' own term, *metabolae*. As he demonstrated, there are several possible ways of modifying the formula of a sieve: we may modify the modulus M or the residue I, or both. These ways have been adequately demonstrated by Squibbs (1996, 57-67) and Gibson (2003, 58-72). However, these two scholars do not make a distinction between formula types and demonstrate transformational possibilities only on the decomposed formula.

Inversion: Sieve

Inversion can be effected simply by reversing the intervallic succession and afterwards re-constructing the sieve accordingly.

Inversion: Periodic formula

On the formula, this can be achieved by altering the residues. In a periodic formula (either decomposed or simplified), e.g. the formula for the *Jonchaies* sieve shown above, we replace all the residues by their negative value and then reduce according to the modulus. Module (17, 5) would then become (17, -5) and finally (17, 12). When this is applied to all modules, the resulting formula produces the inversion:[14]

a) (17, 0) + (17, 1) + (17, 4) + (17, 5) + (17, 7) + (17, 11) + (17, 12) + (17, 16)

b) (17, 0) + (17, -1) + (17, -4) + (17, -5) + (17, -7) + (17, -11) + (17, -12) + (17, -16)

c) (17, 0) + (17, 16) + (17, 13) + (17, 12) + (17, 10) + (17, 6) + (17, 5) + (17, 1)

d) (17, 0) + (17, 1) + (17, 5) + (17, 6) + (17, 10) + (17, 12) + (17, 13) + (17, 16)

Inversion: Inner-periodic formula

To produce the inversion of a sieve from an inner-periodic simplified formula, we replace each *I* by the difference between the highest point of the sieve (*n*) and the final point of the module it belongs to: $n - (R \cdot M + I)$, where *R* is number of repetitions of each module.[15] The inner-periodic formula that produces the inversion of the sieve of *Nekuïa* is

$$24_{16} + 14_{2} + 22_{19} + 31_{22} + 28_{25} + 29_{21} + 19_{2} + 25_{0} + 24_{2} + 26_{19} + 23_{21} + 24_{6} + 30_{19} + 35_{1}) + 29_{6} + 32_{31} + 30_{29} + 26_{15} + 30_{11} + 31_{10}.$$

Cyclic transposition: Sieve

The most straight-forward way of producing a cyclic transposition is by shifting the sieve's intervallic succession to the right or to the left and reducing modularly all points that exceed the period (or range) of the sieve.

Cyclic transposition: Periodic formula

Similar to inversion, we can transpose cyclically a sieve by altering the residues in a periodic formula. If we add the same number to all residues in the formula, we get a transposition of the same degree as the added number. Adding different number to different moduli still gives a cyclic transposition, so long as we add the same number to all residues of the same modulus (see Gibson 2003, 61-3).

Moduli substitution

Whereas affecting the residues (or even the unit of the sieve) merely rearranges the intervallic succession, changing the moduli in a formula produces new structures. As is well known, Xenakis applied this method in *Nomos Alpha*, a process of modulus substitution, which has been termed "generalised Fibonacci process" (Agon *et al*, 2004, 153-5). Xenakis chose a specific modulo multiplication group (Z^*_{18}) and a specific cyclic path of integers within that group (starting with 11 and 13) and taking the numbers in pairs he formed the moduli in each generated formula (see also, Gibson 2003, 79 & Vriend 1981, 54-5).

11, 13, 17, 5, 13, 11, 17, 7, 11, 5, 1, 5, 5, 7, 17, 11, 7, 5, 17 , 13,5 11, 1, 11...

The possible tools of sieve transformation are unlimited. Xenakis' metabolic process for the sieves of *Nomos Alpha* is essentially a purpose-built algorithm (which also involved residue substitution in order to avoid sparse sieves); one can conceive of any sort of mechanisable processes that could produce new versions of sieves. This paper tried to provide the theoretical basis for such endeavours. The central problem of the theory, the redundancy of formulae for a given sieve, is overcome (by either basic number theory or algorithmic processes) through a typology of formulae types. The composer or analyst can choose a formula type according to the information sought or the type of metabolae one desires to achieve. Using Xenakis' famous dichotomy of inside- and outside-time structures, we can say that while the sieve is outside of time, metabolae involve the process of creating new outside-time structures, but also a sketch for the way sieves are employed inside time.

Implementation

In the remainder of this chapter, we describe *sieve.maxpat*, an interactive implementation of Xenakis' sieves, featuring graphical methods of plotting, transforming their contents, and listening to their sonic counterparts. The objective of this implementation is to enable musicians to use sieve-derived structures without prior knowledge of number theory; for those who do, it streamlines the process of doing so, allowing for more rapid visual and auditory exploration of sieve space.

Sieve Logic

The implementation we present here is based on Ariza's implementation in the Python programming language (2005). There are, however, certain significant differences between our approaches.

The prime departure is that his starting point is the formula (what he calls *sieve*) and not the sieve as a series of *points* (which he calls *sieve segment*; Ariza 2005, 46). He distinguishes between *complex* and *simple sieve models*. The simple sieve employs at most two levels of grouping of modules (*residual classes*), where in the inner level there is only intersection, and in the outer outer level there is only union; in a *maximally simple sieve* (equivalent to our simplified formula) modules are combined only by union. As the primary interface, the formula can be handled by *compression*, a process that always results in the production of a maximally simple sieve (a sieve that has not been compressed is *expanded*). There are two forms of compression: *compression by intersection* can be applied to a simple sieve by combining all modules within inner intersection groups into a single module; a complex sieve can be compressed only *by segment*, by generating a sieve segment (a series of points), then resampling the values within this set to produce a maximally simple sieve (Ariza 2005, 42). The first of these to processes is *non-lossy*, whereas in compression by segment "the compressed and expanded sieves will generate identical sieve segments *only for the* z *provided during compression*", the default z-length being 0-99 (Ariza 2005, 42 & 51; emphasis added).

However, it is not clear why compression by intersection is only possible for a simple sieve, with its two levels of grouping, and not for a complex sieve with its multiple levels of grouping; one would naturally assume that it is indeed possible to combine all modules in any intersection group, on any level of nesting. Given that compression by segment is lossy, this is a significant analytical issue. As we can see, starting from opposite sides of the spectrum, the two approaches have to inevitably deal with the same fundamental problem: multiple formulas can produce the same series of points; and therefore, when working on a formal level, introducing steps that involve actual data (sieve segments), formal equivalence is disrupted. In our method, we have proposed two different transitions from the actual level of sieves to the formal level of the logical expression (one prime-arithmetic and one algorithmic) that both provide a unique (albeit one-way) relationship between these two levels.

Functionality

The interface is built within Max/MSP 5,[16] using Grill's *pyext*[17] Max external to integrate Python objects within the Max environment. The entirety of the sieve number logic remains within Python, with graphical I/O performed with Max objects. Between the two lies a layer based upon *pyext*, implemented as another Python class, "pysieve". This layer serves to mediate between the two, acting as the "controller" layer of a Model-View-Controller architecture (Krasner 1988).

The fundamental operation is as follows (depicted in Figure 4):

a) The user enters a sieve *either* as a textual formula (via a text input), *or* as a series of points on the number line (via a graphical display)

b) The user selects whether the formula is to be decomposed or simplified

c) The sieve is displayed graphically (if entered as a textual formula), and represented as an output formula through the algorithmic process of simplification.

The structure of the sieve can then be modified, either textually or graphically, with the results immediately visible. Alongside a number-line plot, the set of integers contained within the sieve is listed for reference.

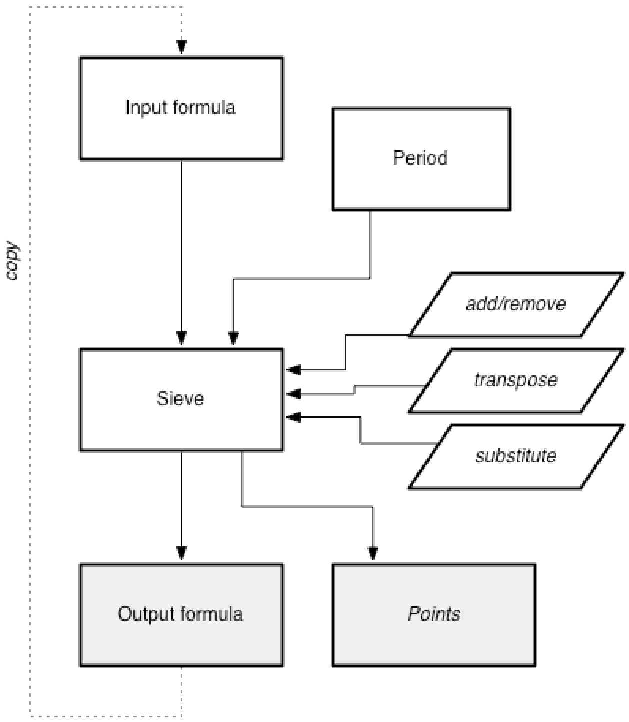

Figure 4: Activities and control flow.

Once a sieve has been constructed, it can then be transformed according to the metabolae described above. At present, transformations include *substitution* and *transposition*.

Finally, it is possible to hear an auditory representation of the sieve's structure through Max/MSP's MIDI output. The points within a sieve can be treated in a number of different fashions:

- as a pitch class, with their numeric indices corresponding to MIDI note numbers; in this case, they are heard in sequence, allowing the composer to treat the contents of a sieve as a non-octave scale. These pitches can either be treated as semitones (as per classic MIDI pitch numbering) or whole tones. A future extension is planned which will allow fractional pitches, for microtonal composition.
- as time intervals, with their indices corresponding to offsets in time; in this case, they are treated as a rhythmic sequence.

If a MIDI keyboard is connected to the software, it can be used to trigger notes within the sieve by generating note-on events: keyboard presses will thus only produce pitches within the sieve's scale. The graphical user interface of the instrument is depicted in Figure 5.

Limitations

Due to the graphical representation being limited by the width of the display window, the number line is limited to a maximum of 88 points. Calculating simplified sieves for large prime moduli can require a very long time, during which time the graphical display will remain frozen.

Obtaining the application

sieve.maxpat is freely available for download and usage. It requires Max/MSP 5 plus a working installation of Python 2.6, with the framework described in Ariza (2005). The software can be obtained from the website of the Xenakis International Symposium:
http://www.gold.ac.uk/ccmc/xenakis-international-symposium/programme/

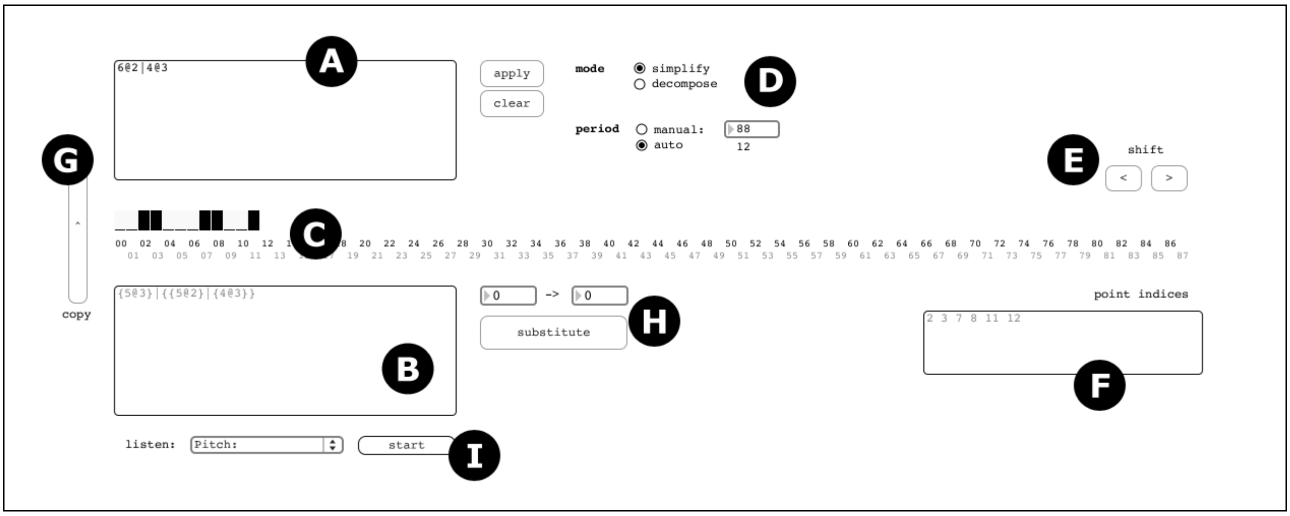

Figure 5: Screenshot of the sieve interface.

Item	Function
A	Formula input, using the notation of Xenakis [(3, 0) + (4, 1)] or Ariza [{ 3@0 | 4@1 }]. Can be validated (with "apply") or cleared.
B	Formula output, minimised based on simplification or decomposition (see D) and relative to the currently-selected period.
C	Sieve number line, with marked points included within the current sieve. If the last user action was from the formula input (A), the points on this line are updated accordingly. Points can be added manually, which will update the formula output (B).
D	Mode selection (simplify vs decompose) and period selection (auto vs manual). Manual period will set the outer periodicity regardless of the formula; automatic mode will set the period to the lowest common multiple of the input formula's moduli.
E	Sieve transposition: shifts all points in the sieve one unit to the left or right.
F	Point indices: a set containing all of the numeric points within the sieve.
G	Copy: copies the formula output to input (reflecting manual changes to the sieve).
H	Substitute: replaces every instance of modulus X with modulus Y in the output formula.
I	Listen: generates musical output based on the current sieve, taking its constituent points as pitch values (MIDI tones or semitones) or time intervals (to generate rhythmic sequences).

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Notes

1 The first published material on Sieve Theory was in Xenakis' article "La voie de la recherche et de la question", in

Preuves 177 (1965), later included in *Musique. Architecture* (Xenakis 1976). The first extended reference was made in "Vers un métamusique" in *La Nef* 29 (1967). This was followed by the ultimate publication of "Sieves", (Xenakis 1990). The two latter articles appeared later as chapters VII and XI of *Formalized Music* (Xenakis 1992).

2 We can only form a union of distinct sets. An example given by Xenakis (1992, 270) involves the union of modules (2, 0) and (6, 0); but (6, 0) is a subset of (2, 0) and therefore the union is redundant.

3 This is what Squibbs terms 'spacing' (see Squibbs 1996, 47).

4 Xenakis used complementation in his former writings only. In (1990) he used only union and intersection.

5 See preface to the score of *Mists*.

6 Since all prime numbers greater than 2 are odd and the sum of all intervals in an intervallic succession equals the period, the number of the intervals must be odd too. Therefore, all prime sieves have an odd number of elements.

7 Xenakis defined the audible range as extending to 11 octaves or 132 semitones (see Xenakis 1976, 67).

8 Note that "modulus" might also refer to the period (P) of the sieve. When appropriate we will distinguish between P and the moduli (M) that make up the modules. Decomposition might be applied to either (as in the periodic and inner-periodic decomposed formulae).

9 After modular reduction of the residues, there can be no intersection of modules that share the same modulus.

10 In the interest of clarity, the residues in long formulae will appear in subscript.

11 In our implementation of this algorithm, we have corrected the redundancy check, as well as the calculation of the LCM of all moduli. Also, unlike Xenakis' description in "Sieves", the program starts checking with $M = 1$ and not 2. The version of the program we used includes Squibbs' corrections (see Squibbs 1996, 291-303).

12 For a detailed analysis of the analytical information provided by the inner-periodic simplified formula, see Exarchos 2008, Chapter 5.

13 After the decomposition of the moduli, the residues are modularly reduced. For example, (26, 17) = (2, 17)·(13, 17) = (2, 1)·(13, 4), because $17 mod 2 \equiv 1$ and $17 mod 13 \equiv 4$.

14 The same procedure would be applied on a formula that involves intersection of moduli whose LCM is the sieve's period, i.e. a decomposed formula.

15 The number of points covered by a module is then $R + 1$. This is slightly different from Xenakis' program, which uses R to denote the number of points each module covers (in our case $R + 1$). In his pre-compositional sketches Xenakis used R to denote the occurrences of a modulus, instead of the number of points covered. This indicates more effectively the contribution of each modulus to the inner-periodic structure of a sieve.

16 http://cycling74.com/products/maxmspjitter/

17 https://github.com/qdot/pyext