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Efficient detection of useful long-range entanglement in imperfect cluster states

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Photonic cluster states are a crucial resource for optical quantum computing. Recently a quantum dot single photon source has been demonstrated to produce strings of photons in a linear cluster state, but high photon loss rates make it impossible to characterize the entanglement generated by conventional methods. We present a benchmarking method for such sources that can be used to demonstrate useful long-range entanglement with currently available collection/detection efficiencies below 1%. Measurement of the polarization state of single photons in different bases can provide an estimate for the three-qubit correlation function $\langle ZXZ \rangle$. This value constrains correlations spanning more than three qubits, which in turn provide a lower bound for the localizable entanglement between any two qubits in the large state produced by the source. Finite localizable entanglement can be established by demonstrating $\langle ZXZ \rangle > \frac{2}{3}$. This result enables photonic experiments demonstrating computationally useful entanglement with currently available technology.

I. INTRODUCTION

Measurement-based quantum computation (MBQC) [1] has become a promising candidate for the most resource-efficient way to build a universal quantum computer. The greatest challenge of MBQC is the generation of a sufficiently large entangled resource state. This step is critical because only certain types of multi-qubit entanglement are known to enable universal quantum computation [2], most prominently cluster state entanglement [3, 4].

Photonic systems have been proposed to generate cluster states [5, 6]. Recently a complete architecture for a linear optical quantum computer has been developed that relies on the probabilistic fusion of many small entangled states into one large cluster state [7]. In this proposal, the generation of the entangled resource state requires only the generation of many maximally entangled three-photon states, at the cost of a significant overhead of single-photon detection measurements and classical information processing. This overhead could be dramatically reduced by developing deterministic “machine gun” sources of photonic cluster states [8, 9].

As experimental work [10] makes progress towards realizing the cluster state source proposed in [8], the challenge of benchmarking such systems comes to the fore. The state produced will certainly be affected by errors and one needs to quantify to what extent the functionality of the state as a resource for MBQC is affected. Since any useful resource state spans many qubits, any benchmarking method with exponential scaling, such as standard quantum state tomography, is out of the question. In fact a benchmarking method for cluster state sources has been presented in [11] which requires a number of single-photon measurements that scales only linearly in the size of the state being benchmarked. We briefly review this method before stating our result.

Localizable entanglement ($LE^{i,j}(\rho)$) can be taken as the figure of merit of a cluster state approximation. It is defined with respect to two qubits i, j in a state ρ of n qubits as the

maximum entanglement between i and j that can be obtained by means of a sequence of single-qubit measurements on the $n - 2$ other qubits in ρ [12]. The maximum is taken over all measurement bases and we take concurrence to be the entanglement measure. The task of benchmarking a cluster state source is accomplished by lower bounding the localizable entanglement of the state that it produces.

Particular triplets of expectation values $\langle B_i \rangle$, $i = 1, 2, 3$ provide a lower bound to the localizable entanglement, as shown in Appendix A. We refer to this bound as the *direct bound* in the following. The operators B_i are tensor products of Pauli operators and their expectation values can be measured as correlations between outcomes of single-qubit measurements in the corresponding Pauli bases. [11] proposes a simple experimental setup to measure these expectation values for photonic systems such as the one proposed in [8]. This benchmarking method, however, is limited by the fact that the time required to measure $\langle B_i \rangle$ increases exponentially with the number of photons in the state to be benchmarked and the inverse of the collection/detection efficiency. Hence only small states can be benchmarked using the direct bound in currently feasible experiments.

The method presented here overcomes this limitation. We find that if the expectation values $\langle Z_{i-1}X_iZ_{i+1} \rangle$ on any three neighboring qubits in a linear state ρ are no smaller than some value $\langle ZXZ \rangle$, then

$$LE^{j,j+k}(\rho) \geq \max\{0, 1 - (k+1)(1 - \langle ZXZ \rangle)\}. \quad (1)$$

Hence measurement of $\langle ZXZ \rangle$ alone can suffice to establish localizable entanglement across many qubits, as illustrated in Fig. 1.

Furthermore, the expectation values $\langle Z_{i-1}X_iZ_{i+1} \rangle$ give information on the usefulness of a state for practical quantum information processing tasks. For instance we find that a linear state ρ with expectation values $\langle Z_{i-1}X_iZ_{i+1} \rangle \geq \langle ZXZ \rangle$ enables a quantum teleportation channel across $n - 2$ qubits of fidelity

$$F_T \geq 1 - \frac{n}{3}(1 - \langle ZXZ \rangle). \quad (2)$$

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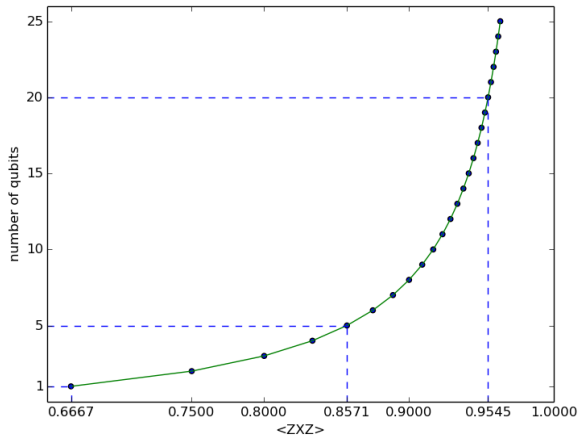


Figure 1. A sufficiently high value of $\langle ZXZ \rangle$ guarantees non-zero LE across many qubits. For instance a value of $\langle ZXZ \rangle = 0.8571$ (0.9545) is sufficient to demonstrate non-zero LE across 5 (20) qubits. The curve shown is obtained by setting (1) to zero and plotting $k - 1$ (number of measured qubits) vs. $\langle ZXZ \rangle$.

In the following we present proofs of the two results (1) and (2) and describe their applicability.

II. THE CLUSTER BACKBONE AND TRANSLATIONAL INVARIANCE

Cluster states can be elegantly defined in the stabilizer formalism [13]. The linear cluster state on n -qubits $|\Psi_{CS}\rangle$, for instance, is the unique eigenstate with eigenvalue $+1$ of the $n - 2$ Pauli operators $K_i = \langle Z_{i-1}X_iZ_{i+1} \rangle$, where the subscripts label any three adjacent qubits, as well as the two operators $K_1 = \langle X_1Z_2 \rangle$ and $K_n = \langle Z_{n-1}X_n \rangle$ at the ends. This collection of n Pauli operators is called the *stabilizer generator*, and the group of all 2^n products of any number of operators in the stabilizer generator is the *stabilizer group* of the linear cluster state. Clearly the cluster state is the $+1$ eigenstate of all operators in the stabilizer group, which we call *stabilizers*. In an imperfect cluster state the expectation values of stabilizers are generally smaller than unity and there may be nonzero expectation values of non-stabilizer operators. We call the collection of expectation values of stabilizers the *cluster state backbone* of a state, because these values determine how similar a state is to the ideal cluster state.

We can furthermore classify stabilizers K according to the number of generators K_i that need to be multiplied to obtain K . We denote a stabilizer operator that is the product of m generators as K^m . Clearly there are $\binom{n}{m}$ stabilizers K^m where $1 \leq m \leq n$.

In the following we make the assumption of *weak translational invariance* (TI) of the state to be characterized. In its strongest form TI requires the reduced density matrix of any number of adjacent qubits to be the same, i.e. independent of where in the n qubit state the segment is found. We

only require the weaker assumption that $\langle K_i \rangle = \langle K_j \rangle \forall i, j$ and denote $\langle K_i \rangle = \langle ZXZ \rangle$ in the following. The assumption of translational invariance is reasonable because the large environments in semiconductor quantum light sources can be assumed to induce mainly Markovian errors, which give rise to TI states. Even the non-Markovian errors modelled in [14] satisfy TI.

III. HOW DOES $\langle ZXZ \rangle$ CONSTRAIN THE CLUSTER BACKBONE?

While generator expectation values $\langle ZXZ \rangle = 1$ constrain the entire cluster backbone to $\langle K^m \rangle = 1$, the values $\langle K^m \rangle$ are restricted to finite intervals when $\langle ZXZ \rangle < 1$. To determine a lower bound on these intervals we consider two commuting multi-qubit Pauli operators P_1 and P_2 . The quantity

$$\begin{aligned} & \text{Tr} \left[\frac{1}{2} (\mathbb{1} - P_1) \frac{1}{2} (\mathbb{1} - P_2) \rho \right] \\ &= \text{Tr} \left[\frac{1}{4} (\mathbb{1} - P_1 - P_2 + P_1 P_2) \rho \right] \end{aligned} \quad (3)$$

gives the joint probability of measuring both P_1 and P_2 with outcome -1 in state ρ and is therefore non-negative. An alternative argument for its non-negativity is that the product of two commuting positive-semidefinite matrices is positive-semidefinite itself.

Setting the r.h.s. of (3) non-negative yields a bound that is fundamental to this work:

$$\langle P_1 P_2 \rangle \geq \langle P_1 \rangle + \langle P_2 \rangle - 1. \quad (4)$$

For the case of the cluster backbone for which only the expectation values of the generators $\langle K_i \rangle = \langle ZXZ \rangle$ are known, it follows that

$$\langle K^m \rangle \geq m(\langle ZXZ \rangle - 1) + 1. \quad (5)$$

IV. THE WC STATE

Imagine a device that is capable of emitting a long string of single photons in a TI state ρ with long-range entanglement potentially similar to that of the cluster state. All that can be measured on ρ is the value of $\langle ZXZ \rangle$. What is the worst-case scenario in terms of long-range entanglement of the state produced, i.e. what is the minimal $\text{LE}^{j, j+k}(\rho)$ consistent with the known $\langle ZXZ \rangle$?

Note that when we consider the segment of the state ρ spanning qubits j to $j+k$, we mean the state on qubits j to $j+k$ that remains after qubits $j-1$ and $j+k+1$ have been measured in the Z basis with outcomes $+1$ (see Fig. 2). Thus the two expectation values $\langle Z_{j-1}X_jZ_{j+1} \rangle$ and $\langle Z_{j+k-1}X_{j+k}Z_{j+k+1} \rangle$ become $\langle X_jZ_{j+1} \rangle$ and $\langle Z_{j+k-1}X_{j+k} \rangle$, respectively, which constitute the boundary stabilizers of a cluster state. This construction is necessary because a segment of a perfect cluster state only becomes a cluster state itself after the boundary

qubits are “clipped off” by Z measurements. For ease of notation, we denote the state of such a n -qubit segment by ρ_n in the following and label the qubits from 1 to n .

Eq. (5) yields a lower bound to any unknown stabilizer expectation value $\langle K^m \rangle$, and certain triplets $\langle B_i \rangle$, $i = 1, 2, 3$, of these expectation values provide a direct bound to $\text{LE}^{1,n}(\rho_n)$. One might now wonder what highest direct bound can be inferred in this way and whether a physical state ρ_n can saturate it. Both these questions are answered by considering a state which we call the *worst-case state* (WC state) on n qubits and define as

$$\rho_n^{WC} = \lambda |C_n\rangle \langle C_n| + \frac{1-\lambda}{n} \sum_{i=1}^n Z_i |C_n\rangle \langle C_n| Z_i, \quad (6)$$

where $|C_n\rangle$ is the linear cluster state on n qubits. The parameter λ is chosen such that ρ_n^{WC} is consistent with a given value of $\langle ZXZ \rangle$, i.e.

$$\lambda = 1 - n \frac{1 - \langle ZXZ \rangle}{2}. \quad (7)$$

The defining property of the WC state is that it saturates all inequalities given by (5):

$$\begin{aligned} \text{Tr}[K^m \rho_n^{WC}] &= \lambda + \frac{1-\lambda}{n} \sum_{i=1}^n \text{Tr}[K^m Z_i |C_n\rangle \langle C_n| Z_i] \\ &= \lambda + \frac{1-\lambda}{n} (n - 2m) \\ &= m(\langle ZXZ \rangle - 1) + 1. \end{aligned} \quad (8)$$

The second line is obtained by noting that K^m anticommutes with m of the Z_i operators, yielding a -1 for m terms in the sum on the r.h.s. of the first line. The third line follows by substituting λ from (7).

Since ρ_n^{WC} is obviously a physical state and saturates all the inequalities of (5), these constraints are in fact a tight bound on the operators in the cluster backbone. Furthermore, ρ_n^{WC} is the state with the lowest possible direct bounds on LE consistent with the given $\langle ZXZ \rangle$, because any other state cannot have expectation values in the cluster backbone smaller than $\text{Tr}[K^m \rho_n^{WC}]$. Certain triplets of these expectation values, however, guarantee a certain value of LE, as shown in Appendix A. Hence no other state can have LE lower than the direct bound that would be measured in ρ_n^{WC} , the value of which is derived in the next section.

V. THE $\langle ZXZ \rangle$ BOUND

A sequence of X or Y measurements on qubits $2, 3, \dots, n-1$ performed on a linear cluster state results in a maximally entangled state on qubits 1 and n . This ideal resultant two-qubit state can be written as

$$\tau_{1,n} = \frac{1}{4} (\mathbb{1} \otimes \mathbb{1} + \sum_{i=1}^3 t_i \sigma_i^1 \otimes \sigma_i^n). \quad (9)$$

Photons	●	●	●	●	●	●	●
Measurement sequence	Z		Y	Y	X		Z
Stabilizer generators	Z	X	Z		Z	...	Z
		Z	X				X
				Z			Z
Surviving stabilizers	Z	Y	Y		X		Z
	Z	X		X	X		Y
		Z	Y	X			X
							Z

Figure 2. Example of a stabilizer description of a measurement sequence that localizes entanglement in a cluster state. The first and last photons (grey dots) are measured out first, such that the state segment has boundary stabilizers XZ and XZ . The state ρ_n describes the $n = 5$ qubits represented by black dots. Out of the entire stabilizer group only the three operators at the bottom commute with all three measurement projectors and therefore remain in the stabilizer group of the resultant state. It is easy to verify that the surviving stabilizers are a product of three, four, and three generators, respectively (top to bottom), such that m_i take the values 3, 4, 3.

For example, a sequence of Y measurements on an ideal linear cluster state with $+1$ outcomes leads to $\sigma_i^1 = \{Y, Z, X\}$, $\sigma_i^n = \{Z, Y, X\}$, and $\vec{t} = (1, 1, 1)$ [11]. We can always take $t_i \geq 0$ by absorbing a minus sign in σ_i^1 or σ_i^n . For states of this form the concurrence is given by [15]

$$C(\tau) = \max\{0, \frac{1}{2}(t_1 + t_2 + t_3 - 1)\}. \quad (10)$$

Given a multi-qubit state ρ , the coefficients t_i can be found as the expectation values of the three cluster state stabilizers that commute with every single-qubit measurement (see Fig. 2 for an example). In a faulty cluster state t_i will take absolute values smaller than one, and measurement of these three expectation values yields the direct bound (see Appendix A) [11]. However, as mentioned in the introduction, the operators with expectation values t_i have large support for long strings of photons produced and are therefore hard to measure.

In ρ_n^{WC} every expectation value takes the minimum value allowed by a given $\langle ZXZ \rangle$, and therefore $C(\tau)$ is minimized for every sequence of measurements and outcomes on qubits $2, 3, \dots, n-1$. In the following we derive the expression for this bound on LE of ρ_n^{WC} .

Each t_i is given by an expectation value $\langle K^{m_i} \rangle$ in ρ_n^{WC} and by (5) can be written as

$$t_i = m_i(\langle ZXZ \rangle - 1) + 1, \quad (11)$$

where m_i gives the number of stabilizer generators contained in the operator K^{m_i} . Using (10) we obtain

$$C(\tau) = \max\{0, \frac{1}{2}(m_1 + m_2 + m_3)(\langle ZXZ \rangle - 1)\}. \quad (12)$$

The number $m_1 + m_2 + m_3$ depends only on the number of single-qubit measurements in the X or Y basis and not on the sequence itself. For instance, a single X measurement leads

to operators containing one, two, and three ZXZ s, while a Y measurement will give three operators each containing two generators. Every further X or Y measurement appends another generator ZXZ to two of the three stabilizers that commute with the measurement sequence, so that

$$m_1 + m_2 + m_3 = 4 + 2m = 4 + 2(n - 2), \quad (13)$$

where m is the number of X or Y measurements and n the number of qubits in ρ_n^{WC} . Hence the LE across $n - 2$ qubits in a state $\rho_n^{WC}(\langle ZXZ \rangle)$ is lower bounded by

$$C(\tau_{1,n}) = \max\{0, 1 - n(1 - \langle ZXZ \rangle)\}. \quad (14)$$

We refer to this value of concurrence as the $\langle ZXZ \rangle$ bound in the following. (1) follows by considering a segment ρ_n of ρ as defined above and relabeling qubit 1, n as qubit j , $j + k$, respectively.

Localizable teleportation channel fidelity

While localizable entanglement certainly is necessary for MBQC, a state of a given LE does not necessarily enable any quantum computation. This discrepancy between entanglement and actual usefulness of a state for quantum information processing tasks has led to the introduction of the fully entangled fraction F as practical measure of quantum information processing significance [16]. The fully entangled fraction (FEF) is defined as the maximum fidelity of a given state ρ with a maximally entangled state $|\phi^+\rangle$ that can be achieved by a local unitary:

$$F(\rho) := \max_U \langle \phi^+ | (\mathbb{1} \otimes U) \rho (\mathbb{1} \otimes U^\dagger) | \phi^+ \rangle. \quad (15)$$

A two-qubit state ρ_T shared between Alice and Bob enables Alice to teleport a quantum state to Bob with a fidelity of

$$F_T = \frac{1}{3}(1 + 2F(\rho_T)) \quad (16)$$

as shown in [17]. Remarkably, entanglement as measured by concurrence is a necessary but not sufficient condition for a nonzero FEF. There is, for instance, a state of both concurrence and FEF equal to 0.5. Many copies of this state can be used to distill a number of Bell pairs, yet a single copy offers no advantage over any separable pure state for quantum teleportation ($F_{|\psi\rangle_{sep}} = 0.5$) [17]. The triplet sum $T := t_1 + t_2 + t_3$ not only lower bounds the concurrence of the state ρ_{t_i} with the expectation values t_i , but also its FEF:

$$F(\rho_{t_i}) \geq \frac{1}{4}(1 + T). \quad (17)$$

Hence the state ρ_n^{WC} with expectation values $\langle ZXZ \rangle$ enables a quantum teleportation channel between the first and n th qubit of fidelity

$$F_T \geq 1 - \frac{n}{3}(1 - \langle ZXZ \rangle). \quad (18)$$

Clearly there is no state that is consistent with $\langle K_i \rangle = \langle ZXZ \rangle$ of lower triplet sum T than ρ_n^{WC} . Hence (18) is a tight lower bound on the teleportation fidelity. This statement is stronger than the $\langle ZXZ \rangle$ bound, because it guarantees that the localizable entanglement present can be used for a single-qubit channel.

Is the $\langle ZXZ \rangle$ bound tight?

The $\langle ZXZ \rangle$ bound rules out the existence of TI states consistent with $\langle ZXZ \rangle$ of lower LE than the value given in (14). The question remains whether this bound is tight, i.e. whether there exists a state which has LE as low as its $\langle ZXZ \rangle$ bound. We believe that the $\langle ZXZ \rangle$ bound is tight because we conjecture that

$$\text{LE}^{1,n}(\rho_n^{WC}(\langle ZXZ \rangle)) = \max\{0, 1 - n(1 - \langle ZXZ \rangle)\}. \quad (19)$$

Appendix B provides strong evidence in support of this conjecture.

VI. COMPARISON TO OTHER CHARACTERIZATION METHODS

A. The direct bound

Measurement of $\langle ZXZ \rangle$ can be accomplished with a simpler experimental setup and fewer photon counts than the direct bound. However the value of LE demonstrated by measuring the direct bound on a source is higher than the LE guaranteed by the $\langle ZXZ \rangle$ bound for this source, except for the unrealistic scenario where the source produces a WC state. Hence there is a trade-off between saving measurement resources and demonstrating a high value of LE.

An interesting scenario to investigate this trade-off is a cluster state source as proposed in [8] subject to uncorrelated Pauli Y -errors on the emitter spin between single-photon emissions. These errors are likely to be the dominant error mechanism in quantum dot implementations of the proposal. Both the direct and the $\langle ZXZ \rangle$ bound can provide two different figures of merit for such a source. Firstly the number of qubits across which the LE is non-zero can be considered; secondly the value of the LE across a fixed number of qubits constitutes a figure of interest. As shown in Fig. 3, the $\langle ZXZ \rangle$ bound is considerably below the direct bound for both quantities. The $\langle ZXZ \rangle$ bound approaches the direct bound as $\langle ZXZ \rangle \rightarrow 1$ (see inset in Fig. 3).

The $\langle ZXZ \rangle$ bound is therefore particularly useful for sources with low emission/detection efficiencies and little decoherence of the emitter spin. In that case no direct bound is available but $\langle ZXZ \rangle$ is close to unity and hence high values of LE across a fixed number of qubits as well as non-zero LE across many qubits can be demonstrated.

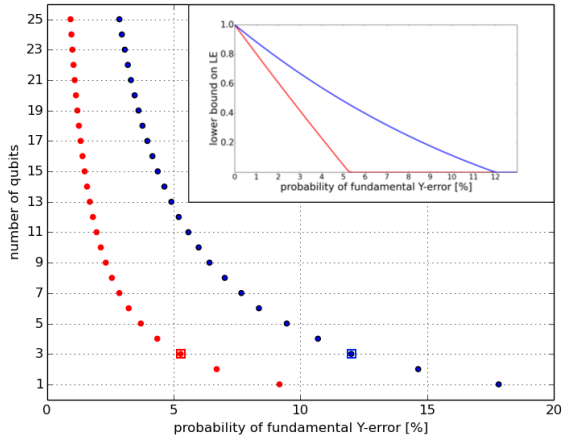


Figure 3. The $\langle ZXZ \rangle$ bound is generally lower than the direct bound. In the main plot the maximum number of qubits across which the direct bound (blue) and the $\langle ZXZ \rangle$ bound (red) guarantee non-zero LE is shown. The x-axis gives the probability of a Pauli Y -error on the emitter spin (fundamental error) before each single-photon emission. The inset shows the value of LE that each bound establishes across three qubits, again plotted against the probability of a fundamental Y -error. The highlighted data points in the main plot correspond to the points in the inset plot where each curve hits zero.

B. Efficient tomography

The idea of characterizing a cluster state approximation by its stabilizer generator expectation values $\langle K_i \rangle$ is not new. In fact a bound on the fidelity of a n -qubit state ρ_n where $\text{Tr}[K_i \rho_n] = \langle K_i \rangle$ with the n -qubit cluster state $|C_n\rangle$ has been derived in [18]:

$$\langle C_n | \rho_n | C_n \rangle \geq 1 - \frac{1}{2} \sum_{i=1}^n (1 - \langle K_i \rangle). \quad (20)$$

For the case of ρ_n satisfying weak TI this bound becomes

$$\langle C_n | \rho_n | C_n \rangle \geq 1 - \frac{n}{2} (1 - \langle ZXZ \rangle). \quad (21)$$

This fidelity bound, however, does not directly give information about the long-range entanglement in a state with $\langle ZXZ \rangle$ approaching one. Clearly two states with the same fidelity with a third, entangled state can themselves have very different entanglement.

Interestingly, the worst-case state ρ_n^{WC} saturates the fidelity bound (21) and thereby sheds new light onto this result, which was obtained by a seemingly unrelated method. The fidelity of a state ρ_n with $|C_n\rangle$ is given by

$$\langle C_n | \rho_n | C_n \rangle = \frac{1}{2^n} \sum_{i=0}^{2^n-1} (-1)^{k_i} \text{Tr}[B_i \rho_n], \quad (22)$$

where B_i are the operators in the cluster backbone, $k_i = \frac{1}{2}(1 + \text{Tr}[B_i |C_n\rangle \langle C_n|])$, and $B_0 = \mathbb{1}$. Using (8) for $\text{Tr}[K^m \rho_n^{WC}]$ we find that ρ_n^{WC} saturates the fidelity bound of (21):

$$\begin{aligned} \langle C_n | \rho_n^{WC} | C_n \rangle &= \frac{1}{2^n} \sum_{m=0}^n \binom{n}{m} [m(\langle ZXZ \rangle - 1) + 1] \\ &= 1 - \frac{n}{2} (1 - \langle ZXZ \rangle), \end{aligned} \quad (23)$$

where we have used the binomial theorem.

The WC state is therefore not only the worst-case scenario for LE, but also the worst-case scenario for fidelity given a value of $\langle ZXZ \rangle$. Moreover since all expectation values in ρ_n^{WC} take on the minimal absolute values allowed by $\langle ZXZ \rangle$, every state that saturates (21) must have the same cluster backbone as ρ_n^{WC} .

VII. CONCLUSION

Our result for the worst-case localizable entanglement in a cluster state approximation substantiates the idea of [18] that an expectation value $\langle ZXZ \rangle$ provides a meaningful benchmark for an experimentally produced cluster state. We have shown that as $\langle ZXZ \rangle$ approaches unity, useful entanglement increases at a rate lower-bounded by (14).

Measurement of an expectation value $\langle ZXZ \rangle$ can be accomplished with the simple setup presented in [11] and emission/detection efficiencies that seem within reach for the experimental implementations of [8], such as the originally proposed quantum dot system or charged NV-centres [18]. The threshold for establishing a cluster state source capable of producing long-range entanglement is therefore lowered to demonstrating $\langle ZXZ \rangle = \frac{2}{3}$, which would demonstrate non-zero localizable entanglement across a single qubit.

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Appendix A: Three expectation values can lower bound the LE

In [11] a bound on the LE was derived given three expectation values and assuming zero entanglement fluctuations [12] (“outcome-independent entanglement”). In the following we show that this bound holds even in the presence of entanglement fluctuations. We only deal with qubits and projective measurements here.

Theorem. *The expectation values of three multi-qubit Pauli operators B_1, B_2, B_3 lower bound the LE as measured by a convex entanglement measure E between two target qubits t_1, t_2 in any state ρ of n qubits as*

$$\text{LE}^{t_1, t_2}(\rho) \geq E(\rho_B). \quad (\text{A1})$$

Here $\rho_B = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} + \langle B_1 \rangle Z \otimes Y + \langle B_2 \rangle Y \otimes Z + \langle B_3 \rangle X \otimes X)$ and $\{B_i\}$ need to satisfy the following:

- There exists a collection of $n - 2$ single-qubit Pauli operators $\{P_k\}$, each acting on one of the n qubits except the target qubits t_1, t_2 , such that $[B_l, P_k] = 0 \forall l, k$. The label k runs from 0 to $n - 2$, while the target qubits are labelled t_1, t_2 .
- $B_3 = B_2 B_1$.
- All three operators B_i have nontrivial support on both qubits t_1 and t_2 , i.e the component of B_i acting on the t_1, t_2 subspace is some two-qubit Pauli operator B_i^t .

Proof. The LE between qubits t_1 and t_2 in a n -qubit state ρ is defined as [12]

$$\text{LE}^{t_1, t_2}(\rho) := \sup_{\{m\}} \sum_s p_s E(\rho_{m, s}^{t_1, t_2}). \quad (\text{A2})$$

Here $\{m\}$ denotes all possible sequences of local measurements on the $n - 2$ qubits other than t_1, t_2 , binary string s

gives the outcomes of such a measurement sequence, and p_s the probability of this outcome. $\rho_{m, s}^{t_1, t_2}$ is the state of qubits t_1, t_2 after a particular measurement sequence m with outcomes s .

The state of qubits t_1, t_2 obtained from ρ by measurement sequence $\{P_k\}$ with outcomes s is written $\rho_{\{P_k\}, s}^{t_1, t_2}$. The average entanglement obtained by $\{P_k\}$ is

$$\sum_s p_s E(\rho_{\{P_k\}, s}^{t_1, t_2}) \quad (\text{A3})$$

and clearly lower bounds $\text{LE}^{t_1, t_2}(\rho)$. Furthermore

$$E(\rho_{\{P_k\}, s}^{i, j}) \geq E(\rho_s), \quad (\text{A4})$$

where $\rho_s := \frac{1}{4}(\mathbb{1} + t_1 B_1^t + t_2 B_2^t + t_3 B_3^t)$ (additional terms cannot decrease entanglement [11]). The coefficients t_i are the expectation values of the corresponding two-qubit Pauli operators B_i in the state $\rho_{\{P_k\}, s}^{t_1, t_2}$.

We now associate strings of $n - 2$ bits with operators Q_q on the $n - 2$ qubits other than t_1, t_2 where a 1 (0) at position k in q means that the component of Q_q acting on qubit k is P_k ($\mathbb{1}_k$). For instance we have $Q_{01101} = \mathbb{1}_1 \otimes P_2 \otimes P_3 \otimes \mathbb{1}_4 \otimes P_5$.

With this notation the coefficients t_i in ρ_s are related to expectation values in the n qubit state ρ (write $\langle \hat{O} \rangle = \text{Tr}[\hat{O}\rho]$) as

$$t_i = \sum_q \frac{(-1)^{s \cdot q}}{2^{n-2} p_s} \langle B_i^t \otimes Q_q \rangle. \quad (\text{A5})$$

The sum runs over all 2^{n-2} bit strings q of length $n - 2$. $s \cdot q$ denotes the modular sum of all those measurement outcomes s_i where $q_i = 1$. One of the terms in the sum of (A5) corresponds to the known expectation value $\langle B_i \rangle = \langle B_i^t \otimes Q_{q_{B_i}} \rangle$.

We can now exploit the convexity of our entanglement measure E and invariance of entanglement under local unitary operations U_s to write

$$\begin{aligned} \text{LE}^{t_1, t_2}(\rho) &\geq \sum_s p_s E(\rho_s) = \sum_s p_s E(U_s \rho_s U_s^\dagger) \\ &\geq E\left(\sum_s p_s U_s \rho_s U_s^\dagger\right). \end{aligned} \quad (\text{A6})$$

The local unitary U_s can be chosen as a tensor product of two Pauli operators such that it anticommutes with two of the three B_i^t . For example if $B_1^t = Z \otimes Y, B_2^t = Y \otimes Z, B_3^t = X \otimes X$ we could choose $U_s = Z \otimes \mathbb{1}$ for a particular s and have $U_s B_1 U_s^\dagger = B_1, U_s B_2 U_s^\dagger = -B_2, U_s B_3 U_s^\dagger = -B_3$. We then associate with every choice of U_s a triplet of binary numbers $f_{1,2,3}(s)$ such that $f_i(s) = 1$ if $\{U_s, B_i^t\} = 0$ and $f_i(s) = 0$ if $[U_s, B_i^t] = 0$. Note that $f_1(s)$ and $f_2(s)$ can be chosen freely, while $f_3(s) = f_1(s) \oplus f_2(s)$. The particular choice of $\{U_s\}$ and associated $f_i(s)$ made here is

$$f_i(s) = q_{B_i} \cdot s, \quad (\text{A7})$$

where q_{B_i} is the binary string giving the sequence of 1 and 0 operators in B_i .

Finally the expectation value of B_i^t for the state $\rho_{mix} := \sum_s p_s U_s \rho_s U_s^\dagger$ can be written using (A5) as

$$\text{Tr}[B_i^t \rho_{mix}] = \sum_s p_s \sum_q \frac{(-1)^{q \cdot s \oplus f_i(s)}}{2^{n-2} p_s} \langle B_i^t \otimes Q_q \rangle. \quad (\text{A8})$$

Substituting (A7) for $f_i(s)$ we find

$$\sum_s (-1)^{(q \oplus q_{B_i}) \cdot s} = \begin{cases} 2^{n-2} & \text{when } q = q_{B_i}, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{A9})$$

Hence the functions $f(s, 1)$ and $f(s, 2)$ can be chosen such that $\rho_{mix} = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} + \langle B_1 \rangle B_1 + \langle B_2 \rangle B_2 + \langle B_3 \rangle B_3)$. Without loss of generality any triplet of two-qubit Pauli operators satisfying the properties required in the theorem can be chosen as $B_1 = Z \otimes Y$, $B_2 = Y \otimes Z$ and $B_3 = X \otimes X$ such that

$$\text{LE}^{t_1, t_2}(\rho) \geq E(\rho_B). \quad (\text{A10})$$

□

Appendix B: Is the $\langle ZXZ \rangle$ bound tight?

We provide three items of evidence in support of our conjecture

$$\text{LE}^{1,n}(\rho_n^{WC}(\lambda)) = \max\{0, 1 - n(1 - \langle ZXZ \rangle)\} \\ = \max\{0, 2\lambda - 1\}. \quad (\text{B1})$$

Firstly we show that all equatorial measurement sequences yield entanglement no higher than (14); secondly we show analytically that the claim holds for ρ_4^{WC} ; and finally we present numerics for ρ_7^{WC} .

1. Equatorial measurement sequences

Lemma. Given $\rho_n^{WC}(\lambda)$, no sequence of equatorial measurements can produce a two-qubit state $\rho_{1,n}$ with $E(\rho_{1,n}) > 2\lambda - 1$.

Proof. We first derive a form for the two-qubit state $|\phi_{1,n}\rangle$ resulting from a sequence of equatorial measurements \mathcal{M}_e on qubits 2 to $n-1$ in the n -qubit cluster state $|C_n\rangle$. An equatorial measurement projects onto either of the states $|\varphi, s = 0, 1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^s e^{i\varphi} |1\rangle)$. Using the expression

$$(|\varphi, s\rangle \langle \varphi, s|_A \otimes \mathbb{1}_B) S_{AB} (|\Psi\rangle_A \otimes |+\rangle_B) \\ = |\varphi, s\rangle_A \otimes X^s H(\varphi) |\Psi\rangle_B, \quad (\text{B2})$$

where S_{AB} is the controlled phase gate and

$$H(\varphi) = \begin{pmatrix} 1 & e^{i\varphi} \\ 1 & -e^{i\varphi} \end{pmatrix}, \quad (\text{B3})$$

we find that

$$|\phi_{1,n}\rangle = \mathcal{N} \langle \vec{\varphi}, \vec{s} | C_n \rangle = (\mathbb{1}_1 \otimes U_n) |\phi^+\rangle_{1,n}. \quad (\text{B4})$$

Here $|\phi^+\rangle$ denotes a maximally entangled state, \mathcal{N} the normalization factor,

$$|\vec{\varphi}, \vec{s}\rangle = |\varphi_2, s_2\rangle_2 |\varphi_3, s_3\rangle_3 \cdots |\varphi_{n-1}, s_{n-1}\rangle_{n-1}, \quad (\text{B5})$$

and

$$U_n = X^{s_{n-1}} H(\varphi_{n-1}) \cdots X^{s_3} H(\varphi_3) X^{s_2} H(\varphi_2). \quad (\text{B6})$$

Without loss of generality we can take $s_i = 0$ when writing U_n in the following.

Furthermore we find that

$$\langle \vec{\varphi}, \vec{s} | Z_1 | C_n \rangle \propto (\mathbb{1}_1 \otimes U_n X) |\phi^+\rangle_{1,n} \equiv |\phi_{1,n}^{Z_1}\rangle \\ \langle \vec{\varphi}, \vec{s} | Z_j | C_n \rangle \propto (\mathbb{1}_1 \otimes U_n^j) |\phi^+\rangle_{1,n} \equiv |\phi_{1,n}^{Z_j}\rangle \\ \langle \vec{\varphi}, \vec{s} | Z_n | C_n \rangle \propto (\mathbb{1}_1 \otimes (ZU)_n) |\phi^+\rangle_{1,n} \equiv |\phi_{1,n}^{Z_n}\rangle, \quad (\text{B7})$$

where U_n^j is given by U_n from (B6) with $s_j = 1$ and $s_i = 0 \forall i \neq j$. The state $\rho_{1,n}$ resulting from \mathcal{M}_e on ρ_n^{WC} is therefore given by

$$\rho_{1,n} = \mathcal{N} \langle \vec{\varphi}, \vec{s} | \rho_n^{WC} | \vec{\varphi}, \vec{s} \rangle \\ = \lambda |\phi_{1,n}\rangle \langle \phi_{1,n}| + \frac{1-\lambda}{n} |\phi_{1,n}^{Z_1}\rangle \langle \phi_{1,n}^{Z_1}| \\ + \frac{1-\lambda}{n} \sum_{j=2}^{n-1} |\phi_{1,n}^{Z_j}\rangle \langle \phi_{1,n}^{Z_j}| + \frac{1-\lambda}{n} |\phi_{1,n}^{Z_n}\rangle \langle \phi_{1,n}^{Z_n}|.$$

We now observe that for $1 \leq i \leq n$

$$C(\lambda |\phi_{1,n}\rangle \langle \phi_{1,n}| + (1-\lambda) |\phi_{1,n}^{Z_i}\rangle \langle \phi_{1,n}^{Z_i}|) = 2\lambda - 1, \quad (\text{B8})$$

as $|\phi_{1,n}\rangle \langle \phi_{1,n}|$ and $|\phi_{1,n}^{Z_i}\rangle \langle \phi_{1,n}^{Z_i}|$ are orthogonal, maximally entangled states. Therefore, by convexity of concurrence,

$$E(\rho_{1,n}) \leq 2\lambda - 1. \quad (\text{B9})$$

□

2. Analytics for ρ_4^{WC}

We derive an exact expression for the concurrence of the two-qubit state resulting from two measurements in the X - Z plane on the middle qubits in the four-qubit WC state. The result substantiates our conjecture that general measurement sequences on ρ_n^{WC} cannot reach higher entanglement than equatorial ones.

The result of a projective measurement on qubit i in the X - Z plane may be written as $|\theta_i\rangle = \cos \frac{\theta_i}{2} |0\rangle + \sin \frac{\theta_i}{2} |1\rangle$. We perform measurements on qubits two and three of ρ_4^W , with outcomes parametrized by θ_2 and θ_3 respectively. This yields the two qubit-state

$$\rho_{1,4} = \frac{1}{4} (\mathbb{1} \otimes \mathbb{1} + \vec{r} \cdot \vec{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{s} \cdot \vec{\sigma} + \sum_{i,j=1}^3 T_{ij} \sigma_i \otimes \sigma_j),$$

which has components

$$\vec{r} = \frac{1+\lambda}{2} \begin{pmatrix} \cos \theta_2 \\ 0 \\ \sin \theta_2 \cos \theta_3 \end{pmatrix}, \quad \vec{s} = \frac{1+\lambda}{2} \begin{pmatrix} \cos \theta_3 \\ 0 \\ \sin \theta_3 \cos \theta_2 \end{pmatrix},$$

$$T = \begin{pmatrix} \lambda \cos \theta_2 \cos \theta_3 & 0 & \lambda \sin \theta_3 \\ 0 & (2\lambda - 1) \sin \theta_2 \sin \theta_3 & 0 \\ \lambda \sin \theta_2 & 0 & 0 \end{pmatrix}.$$

The entanglement of a state is invariant under local unitary operations, $\rho_{1,4} \rightarrow (U_1 \otimes U_4) \rho_{1,4} (U_1^\dagger \otimes U_4^\dagger)$. This corresponds to the transformations $\vec{r} \rightarrow O_1 \vec{r}$, $\vec{s} \rightarrow O_4 \vec{s}$, $T \rightarrow O_1 T O_4^T$, where O_1 and O_4 are orthogonal matrices [19]. By choosing O_1 and O_4 that achieve a signed singular value decomposition of T , we perform local unitary operations on $\rho_{1,4}$ that correspond to the transformation

$$\rho_{1,4} \rightarrow \frac{1}{4} \begin{pmatrix} N_+(1+\lambda) & 0 & 0 & S(1-3\lambda) \\ 0 & 1-\lambda & S(\lambda-1) & 0 \\ 0 & S(\lambda-1) & 1-\lambda & 0 \\ S(1-3\lambda) & 0 & 0 & N_-(1+\lambda) \end{pmatrix},$$

where $S = \sin \theta_2 \sin \theta_3$ and $N_\pm = 1 \pm \sqrt{1-S^2}$.

The above density matrix is manifestly in the form of an X -state [20]. The concurrence of such a state is a simple function of the density matrix elements, which for us gives

$$C(\rho_{1,4}) = \max\left\{0, \frac{1}{2}(3\lambda-1)S + \frac{1}{2}(\lambda-1)\right\}. \quad (\text{B10})$$

The entanglement that is localized by measurements in the X - Z plane on ρ_4^W is therefore clearly maximized by the equatorial measurement sequence, which gives $\sin \theta_2 = \sin \theta_3 = 1$ and hence $S = 1$.

3. Numerics for ρ_7^{WC}

We have not found an analytic form for $\text{LE}^{1,n}(\rho_n^{WC})$ when $n > 4$. To further investigate whether our conjecture $\text{LE}^{1,n}(\rho_n^{WC}) = 2\lambda - 1$ holds we therefore perform a numerical optimization.

Using a Nelder-Mead simplex algorithm the following optimization was carried out:

$$\max_{\vec{\theta}, \vec{\varphi}} C(\mathcal{N} \langle \vec{\theta}, \vec{\varphi} | \rho_7^W | \vec{\theta}, \vec{\varphi} \rangle), \quad (\text{B11})$$

where \mathcal{N} gives the normalization factor and

$$|\vec{\theta}, \vec{\varphi}\rangle = \otimes_{i=2}^6 \left(\sin \frac{\theta_i}{2} |0\rangle_i + e^{i\varphi_i} \cos \frac{\theta_i}{2} |1\rangle_i \right). \quad (\text{B12})$$

The results for six different values of λ are shown in Fig. 4. We find that the optimal measurement angles are $\theta_i = \frac{\pi}{2}$ and that φ_i is arbitrary. This yields a value of concurrence $C(\rho_{1,7}) = 2\lambda - 1$, again providing evidence for the conjecture $\text{LE}^{1,n}(\rho_n^{WC}) = 2\lambda - 1$.

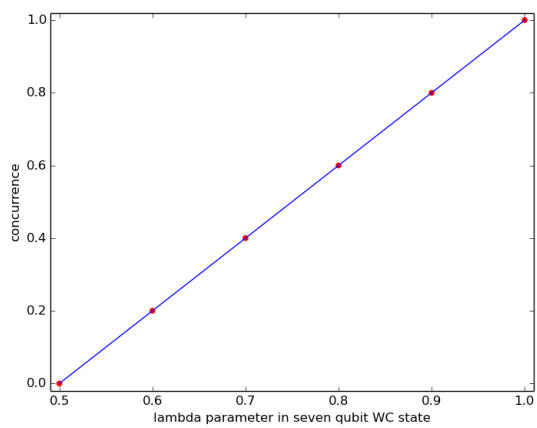


Figure 4. Numerical optimization of the concurrence of the two-qubit state resulting from five single-qubit measurements on ρ_7^W . The red dots show the maximum entanglement obtained while the blue line gives the value $2\lambda - 1$ of the $\langle ZXZ \rangle$ bound.