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Lévy-Vasicek Models and the Long-Bond Return Process

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The classical derivation of the well-known Vasicek model for interest rates is reformulated in terms of the associated pricing kernel. An advantage of the pricing kernel method is that it allows one to generalize the construction to the Lévy-Vasicek case, avoiding issues of market incompleteness. In the Lévy-Vasicek model the short rate is taken in the real-world measure to be a mean-reverting process with a general one-dimensional Lévy driver admitting exponential moments. Expressions are obtained for the Lévy-Vasicek bond prices and interest rates, along with a formula for the return on a unit investment in the long bond, defined by $L_t = \lim_{T \rightarrow \infty} P_{tT}/P_{0T}$, where P_{tT} is the price at time t of a T -maturity discount bond. We show that the pricing kernel of a Lévy-Vasicek model is uniformly integrable if and only if the long rate of interest is strictly positive.

I. PRICING KERNELS

The Vasicek model (Vasicek 1977) is one of the oldest and most well-studied models in the mathematical finance literature, and one might think that there is very little that is new that can be said about it. But it turns out that there are some surprising features of the Vasicek model relating to the long rate of interest that are very suggestive when it comes to modelling long term interest rates in general.

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Time 0 denotes the present. The probability measure \mathbb{P} is the physical measure, and $\{\mathcal{F}_t\}$ represents the flow of market information. We introduce an appropriate unit of account, and for T and t such that $0 \leq t < T$ we let P_{tT} denote the value at time t of a discount bond that pays out one unit of account at maturity T . In what follows we use a pricing kernel method to construct the Vasicek model. This is not the way in which the Vasicek model is usually presented in the literature. Nevertheless, the pricing kernel approach is very effective. In particular, the pricing kernel formulation of the classical Vasicek model leads us directly to a construction of the corresponding Lévy-Vasicek model, extending results of Cairns (1999), Eberlein & Raible (1999), Norberg (2004), and others.

We begin with a few remarks about pricing kernels and then turn to the case of the Vasicek model. Let us recall briefly how pricing kernels work in the elementary geometric Brownian motion (GBM) model for asset prices. We fix a Brownian motion $\{W_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, and take it to be adapted to $\{\mathcal{F}_t\}$. The GBM model is characterized by the specification of a pricing kernel along with a collection of one or more so-called investment-grade assets. We assume for simplicity that the assets pay no dividends over the time horizon considered. The idea of an investment-grade asset is that it should offer a positive excess rate of return above the interest rate. Ordinary stocks and bonds are in this sense investment-grade, whereas put options are not.

For the pricing kernel in the GBM model we assume that we have an expression of the form

$$\pi_t = e^{-rt} e^{-\lambda W_t - \frac{1}{2}\lambda^2 t}, \quad (1)$$

where r is the interest rate, and $\lambda > 0$ is a risk aversion parameter. We require that the product of the pricing kernel and the asset price should be a \mathbb{P} -martingale. Let us suppose therefore that for some $\beta \geq -\lambda$ the product takes the form

$$\pi_t S_t = S_0 e^{\beta W_t - \frac{1}{2}\beta^2 t}, \quad (2)$$

where S_t denotes the value of the asset at time t . For a typical investment-grade asset in the GBM model we thus have

$$S_t = S_0 e^{(r+\lambda\sigma)t} e^{\sigma W_t - \frac{1}{2}\sigma^2 t}, \quad (3)$$

where $\sigma = \beta + \lambda$. The term $\lambda\sigma$ is called the risk premium or excess rate of return, and it is positive under the assumptions we have made. The idea of a “pricing kernel” dates back to the 1970s, and is used for example by Ross (1978). The alternative term “market kernel” is used by Garman (1976). Authors have employed a variety of terms for essentially the same concept. Economists speak of the “marginal rate of substitution”. The term “state price density” appears in Dothan & Williams (1978). The term “stochastic discount factor” is used in Cox & Martin (1983). The term “state price deflator” is used in Duffie (1992).

Pricing kernel models are discussed in detail in, for example, Cochrane (2005) and Hunt & Kennedy (2004). If the risky asset under consideration is a European-style derivative whose terminal payoff is H_T , then the value of the derivative at time $t < T$ is given by

$$H_t = \frac{1}{\pi_t} \mathbb{E}_t[\pi_T H_T]. \quad (4)$$

In particular, if the derivative pays one unit of account so that $H_T = 1$, then we recover the pricing formula for a discount bond, given by

$$P_{tT} = \frac{1}{\pi_t} \mathbb{E}_t[\pi_T]. \quad (5)$$

We refer to the process $\{n_t\}_{t \geq 0}$ defined by $n_t = 1/\pi_t$ as the “natural numeraire” (Flesaker & Hughston 1997), or “growth-optimal portfolio”. It serves as a benchmark, relative to which other non-dividend-paying assets are martingales. As an example of derivative pricing in the GBM model, one can consider the valuation of a digital put on the natural numeraire, with unit notional, strike κ , and maturity T . In this case we have

$$H_T = \mathbf{1}\{n_T < \kappa\}, \quad (6)$$

where $\mathbf{1}\{\cdot\}$ denotes the indicator function. Using the pricing kernel (1), a straightforward calculation gives

$$H_0 = e^{-rT} N \left[\frac{\log(e^{-rT} \kappa) + \frac{1}{2}\lambda^2 T}{\lambda T^{1/2}} \right], \quad (7)$$

where $N[\cdot]$ is the normal distribution function. We mention the example of a digital put on the natural numeraire because it turns out to be relevant later in our consideration of the uniform integrability of the pricing kernel.

II. VASICEK PRICING KERNEL

We shall extend the geometric Brownian motion model by keeping the risk-aversion level constant, but allowing the interest rate to be stochastic. The pricing kernel can then be expressed in the form

$$\pi_t = \exp \left[- \int_0^t r_s ds - \lambda W_t - \frac{1}{2} \lambda^2 t \right]. \quad (8)$$

In the Vasicek model, the short rate process $\{r_t\}_{t \geq 0}$ is taken to be a mean-reverting process of the Ornstein-Uhlenbeck (OU) type, satisfying

$$dr_t = k(\theta - r_t)dt - \sigma dW_t. \quad (9)$$

Here k , θ , and σ denote the mean reversion rate, the mean reversion level, and the absolute volatility of the short rate. We choose σ to be positive. The minus sign in front of σ in the dynamics of the interest rate is a convention that ensures that the discount bond volatility is positive. Note that the volatility parameter here has units of $T^{-3/2}$, in contrast with the σ of the GBM model, which has units of $T^{-1/2}$. The initial value of the interest rate is r_0 . The dynamical equation (9) can then be solved by use of an integrating factor to give

$$r_t = \theta + (r_0 - \theta) e^{-kt} - \sigma \int_0^t e^{k(s-t)} dW_s. \quad (10)$$

To determine the pricing kernel we require an expression for the integrated short rate,

$$I_t = \int_0^t r_s ds. \quad (11)$$

Substitution of (10) into (11) gives

$$I_t = \theta t + \frac{1}{k} (1 - e^{-kt}) (r_0 - \theta) - \sigma \int_{s=0}^t \int_{u=0}^s e^{k(u-s)} dW_u ds. \quad (12)$$

The double integral can be rearranged according to the scheme

$$\int_{s=0}^t \int_{u=0}^s e^{k(u-s)} dW_u ds = \int_{u=0}^t \int_{s=u}^t e^{k(u-s)} ds dW_u = \frac{1}{k} \int_0^t (1 - e^{k(u-t)}) dW_u, \quad (13)$$

from which it follows that

$$I_t = \theta t + \frac{1}{k} (1 - e^{-kt}) (r_0 - \theta) - \frac{\sigma}{k} \int_0^t (1 - e^{k(u-t)}) dW_u. \quad (14)$$

On account of (10) we can then replace the stochastic integral above with an expression involving the short rate to obtain

$$I_t = \theta t + \frac{1}{k} (r_0 - r_t) - \frac{\sigma}{k} W_t. \quad (15)$$

It follows that the Vasicek pricing kernel can be expressed in the form

$$\pi_t = \exp \left[- \left(\theta + \frac{1}{2} \lambda^2 \right) t + \left(\frac{\sigma}{k} - \lambda \right) W_t - \frac{1}{k} (r_0 - r_t) \right]. \quad (16)$$

Note the appearance of the “naked” W_t in the formula for the pricing kernel. Often it is said that the Vasicek model has a single state variable, the short rate. This statement is a little misleading. For while it is true, as we shall see shortly, that the price at time t of a T -maturity discount bond depends only on the state variable r_t insofar as its stochasticity is concerned, the pricing kernel depends at time t on a pair of state variables, namely, r_t and W_t . Likewise, the value of a unit-initialized money market account

$$B_t = \exp \int_0^t r_s ds \quad (17)$$

depends on both r_t and W_t . We take the view that to specify a financial model one needs to give the price processes for the basic assets of the model, together with the pricing kernel process. In the case of an interest rate model this means giving the processes for discount bonds of all maturities, the money market account, and the pricing kernel. Thus, the pricing kernel is part of the model, not a secondary object that one works out in some way from the specification of the processes of basic assets.

An alternative expression for the Vasicek pricing kernel, also useful for our purposes, is obtained by substituting (14) into (8). Then we have

$$\pi_t = \exp \left[- \left(\theta + \frac{1}{2} \lambda^2 \right) t - \frac{1}{k} (1 - e^{-kt}) (r_0 - \theta) + \int_0^t \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-t)} \right) dW_u \right]. \quad (18)$$

In this case, it seems that we have succeeded in specifying the pricing kernel in terms of a single state variable, namely, the value of the stochastic integral appearing in the third term on the right side of (18). This is indeed so, but it is distinct from the short rate, so the model as a whole requires two state variables.

III. DISCOUNT BONDS

We proceed to derive an expression for P_{tT} . In the derivation we find it convenient to work with logarithms rather than exponentials. Thus, instead of (18) we write

$$\log \pi_t = - \left(\theta + \frac{1}{2} \lambda^2 \right) t - \frac{1}{k} (1 - e^{-kt}) (r_0 - \theta) + \int_0^t \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-t)} \right) dW_u. \quad (19)$$

It follows that

$$\begin{aligned} \log \pi_T = & - \left(\theta + \frac{1}{2} \lambda^2 \right) T - \frac{1}{k} (1 - e^{-kT}) (r_0 - \theta) + \int_0^t \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right) dW_u \\ & + \int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right) dW_u, \end{aligned} \quad (20)$$

and hence for $t < T$ we have

$$\begin{aligned} \mathbb{E}_t[\pi_T] = & \exp \left[- \left(\theta + \frac{1}{2} \lambda^2 \right) T - \frac{1}{k} (1 - e^{-kT}) (r_0 - \theta) + \int_0^t \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right) dW_u \right] \\ & \times \mathbb{E}_t \exp \left[\int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right) dW_u \right]. \end{aligned} \quad (21)$$

It is a standard result that for any Borel function $\{\alpha_t\}$ satisfying

$$\int_t^T \alpha_u^2 du < \infty \quad (22)$$

we have

$$\mathbb{E}_t \left[\exp \left(\int_t^T \alpha_u dW_u \right) \right] = \exp \left(\frac{1}{2} \int_t^T \alpha_u^2 du \right). \quad (23)$$

As a consequence, we obtain

$$\mathbb{E}_t \exp \left[\int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right) dW_u \right] = \exp \left[\frac{1}{2} \int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right)^2 du \right]. \quad (24)$$

Therefore, by (5) we have

$$\begin{aligned} \log P_{tT} = & - \left(\theta + \frac{1}{2} \lambda^2 \right) (T - t) - \frac{1}{k} (e^{-kt} - e^{-kT}) (r_0 - \theta) \\ & + \frac{1}{2} \int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right)^2 du + \frac{\sigma}{k} (1 - e^{k(t-T)}) \int_0^t e^{k(u-t)} dW_u. \end{aligned} \quad (25)$$

By use of (10), in the final term above we can write

$$\sigma \int_0^t e^{k(u-t)} dW_u = \theta + (r_0 - \theta) e^{-kt} - r_t. \quad (26)$$

Then the terms in (25) involving $r_0 - \theta$ cancel, and we are left with the following:

$$\log P_{tT} = - \left(\theta + \frac{1}{2} \lambda^2 \right) (T - t) + \frac{1}{2} \int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right)^2 du + \frac{1}{k} (1 - e^{k(t-T)}) (\theta - r_t). \quad (27)$$

Thus, we have isolated the dependence of P_{tT} on the state variable r_t . Next, to work out the middle term on the right we note that for a, b constant we have

$$\int_t^T (a - be^{ku})^2 du = a^2(T - t) - \frac{2ab}{k} (e^{kT} - e^{kt}) + \frac{1}{2} \frac{b^2}{k} (e^{2kT} - e^{2kt}). \quad (28)$$

Hence,

$$\begin{aligned} \int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right)^2 du = & \left(\frac{\sigma}{k} - \lambda \right)^2 (T - t) - 2 \left(\frac{\sigma^2}{k^3} - \frac{\lambda\sigma}{k^2} \right) (1 - e^{k(t-T)}) \\ & + \frac{1}{2} \frac{\sigma^2}{k^3} (1 - e^{2k(t-T)}). \end{aligned} \quad (29)$$

Inserting this expression into (27), we see that the terms involving λ^2 cancel, and after some simplification we obtain the following expression for the value of a T -maturity discount bond:

$$P_{tT} = \exp \left[-R_\infty(T - t) + \frac{1}{k} (1 - e^{k(t-T)}) (R_\infty - r_t) - \frac{1}{4} \frac{\sigma^2}{k^3} (1 - e^{k(t-T)})^2 \right], \quad (30)$$

where

$$R_\infty = \theta + \frac{\lambda\sigma}{k} - \frac{1}{2} \frac{\sigma^2}{k^2}. \quad (31)$$

The significance of R_∞ is that it represents the asymptotic bond yield, or exponential long rate of interest, defined by

$$R_\infty = - \lim_{T \rightarrow \infty} \frac{1}{T-t} \log P_{tT}. \quad (32)$$

The fact that R_∞ does not depend on t is characteristic of interest rate models for which the tail of the discount function is exponential, and can be viewed as a manifestation of the so-called DIR theorem (Dybvig, Ingersoll & Ross 1996, Hubalek, Klein & Teichmann 2002, Goldammer & Schmock 2012, Kardaras & Platen 2012, Brody & Hughston 2016).

IV. UNIFORM INTEGRABILITY OF PRICING KERNEL

We remark on a curious feature of the Vasicek model that apparently has not been noted previously, namely, that $R_\infty > 0$ if and only if the pricing kernel is uniformly integrable. Before we establish this result and its generalization to the Lévy-Vasicek model, we discuss some financial aspects of the uniform integrability of the pricing kernel.

We recall (Williams 1991) that a collection \mathcal{C} of random variables is said to be uniformly integrable (UI) if for every $\epsilon > 0$ there exists a $\delta \geq 0$ such that for all $X \in \mathcal{C}$ we have

$$\mathbb{E}[|X| \mathbf{1}\{|X| > \delta\}] < \epsilon. \quad (33)$$

An equivalent way of expressing the UI condition on \mathcal{C} is then

$$\lim_{\delta \rightarrow \infty} \sup_{X \in \mathcal{C}} \mathbb{E}[|X| \mathbf{1}\{|X| > \delta\}] = 0. \quad (34)$$

The limit on the left side of (34) exists since $\sup_{X \in \mathcal{C}} \mathbb{E}[|X| \mathbf{1}\{|X| > \delta\}]$ is decreasing in δ and bounded from below by zero. A random process $\{X_t\}_{t \geq 0}$ is thus said to be UI if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\mathbb{E}[|X_t| \mathbf{1}\{|X_t| > \delta\}] < \epsilon$ for all $t \geq 0$, or equivalently

$$\lim_{\delta \rightarrow \infty} \sup_t \mathbb{E}[|X_t| \mathbf{1}\{|X_t| > \delta\}] = 0. \quad (35)$$

For a pricing kernel we can drop the absolute value sign, and the UI condition is that for every $\epsilon > 0$ there should exist a $\delta \geq 0$ such that $\mathbb{E}[\pi_t \mathbf{1}\{\pi_t > \delta\}] < \epsilon$ for all $t \geq 0$, or

$$\lim_{\delta \rightarrow \infty} \sup_t \mathbb{E}[\pi_t \mathbf{1}\{\pi_t > \delta\}] = 0. \quad (36)$$

Alternatively, the UI condition can be imposed by requiring that for every $\epsilon > 0$ there should exist a $\kappa > 0$ such that

$$\mathbb{E}[\pi_t \mathbf{1}\{n_t < \kappa\}] < \epsilon \quad (37)$$

for all $t \geq 0$. Here again we have introduced the natural numeraire $\{n_t\}_{t \geq 0}$ defined by $n_t = 1/\pi_t$, and we have set $\kappa = 1/\delta$. But the left side of (37) is the price at time 0 of a European-style digital put option on the natural numeraire with strike κ and maturity t . Thus we have the following:

Proposition 1 *A pricing kernel is uniformly integrable if and only if for any price level $\epsilon > 0$ there exists a strike $\kappa > 0$ such that the value of a digital put option on the natural numeraire is less than ϵ for all maturities.*

It seems intuitively plausible, and hence it is tempting to conjecture, that every economically admissible pricing kernel should have the UI property. For example, if the pricing kernel is a type- D potential (Hunt & Kennedy 2004, Rogers 1997, Rutkowski 1997), then it is UI (Meyer 1966). We proceed by establishing the following for the Vasicek model.

Proposition 2 *If $R_\infty > 0$ then $\{\pi_t\}$ is uniformly integrable.*

Proof. We shall use an \mathcal{L}^p test. We say that a collection \mathcal{C} of random variables is bounded in \mathcal{L}^p if there exists a constant $\gamma > 0$ such that $\mathbb{E}[|X|^p] < \gamma$ for all $X \in \mathcal{C}$. Now, if $p > 1$ and $x \geq \delta > 0$ for $x, \delta \in \mathbb{R}$, then clearly $x \leq \delta^{1-p}x^p$. It follows that if \mathcal{C} is bounded in \mathcal{L}^p then for all $X \in \mathcal{C}$ it holds that

$$\mathbb{E}[|X|\mathbf{1}\{|X| > \delta\}] \leq \delta^{1-p}\mathbb{E}[|X|^p\mathbf{1}\{|X| > \delta\}] < \gamma\delta^{1-p}. \quad (38)$$

Thus given any $\epsilon > 0$ if we set

$$\delta = \left(\frac{\epsilon}{\gamma}\right)^{1/(1-p)} \quad (39)$$

then we have constructed a δ such that (33) holds for all $X \in \mathcal{C}$. Therefore, if a collection of random variables is bounded in \mathcal{L}^p for some $p > 1$, then it is UI. It follows that a sufficient condition for the pricing kernel to be UI is that there should exist a $p > 1$ and a $\gamma > 0$ such that $\mathbb{E}[\pi_t^p] < \gamma$ for all t . Now, a calculation starting with (18) gives

$$\begin{aligned} \log \mathbb{E}[\pi_t^p] &= -p \left[\theta + \frac{1}{2}\lambda^2 - p \frac{1}{2} \left(\frac{\sigma}{k} - \lambda \right)^2 \right] t \\ &\quad + \frac{p}{k} \left[\theta - r_0 - p \frac{\sigma}{k} \left(\frac{\sigma}{k} - \lambda \right) \right] (1 - e^{-kt}) + \frac{p^2 \sigma^2}{4k^3} (1 - e^{-2kt}). \end{aligned} \quad (40)$$

The second and third terms on the right are bounded, so our goal is to show that if $R_\infty > 0$ then there exists a value of $p > 1$ such that the coefficient of t in the first term on the right in (40) is less than or equal to zero. Suppose therefore that $R_\infty > 0$. Then we have

$$\theta > \frac{1}{2} \frac{\sigma^2}{k^2} - \frac{\lambda\sigma}{k}. \quad (41)$$

Completing the square on the right, we get

$$\theta + \frac{1}{2}\lambda^2 > \frac{1}{2} \left(\frac{\sigma}{k} - \lambda \right)^2. \quad (42)$$

Therefore if we set

$$p = \frac{\theta + \frac{1}{2}\lambda^2}{\frac{1}{2} \left(\frac{\sigma}{k} - \lambda \right)^2}, \quad (43)$$

then it follows that $p > 1$ and that the first term on the right side of (40) vanishes. Since the other two terms are bounded, this shows that if $R_\infty > 0$ then the \mathcal{L}^p test is satisfied, and the pricing kernel is UI. \square

Proposition 3 *If $R_\infty < 0$ then $\{\pi_t\}$ is not uniformly integrable.*

Proof. It is well known (Williams 1991) that if a collection of random variables is UI then it is bounded in \mathcal{L}^1 . For suppose that \mathcal{C} has the property that for every $\epsilon > 0$ there exists a $\delta \geq 0$ such that (33) holds for all $X \in \mathcal{C}$. Then there exists a constant δ_1 such that $\mathbb{E}[|X|\mathbb{1}\{|X| > \delta_1\}] < 1$ for all $X \in \mathcal{C}$, and therefore

$$\mathbb{E}[|X|] = \mathbb{E}[|X|\mathbb{1}\{|X| > \delta_1\}] + \mathbb{E}[|X|\mathbb{1}\{|X| \leq \delta_1\}] < 1 + \delta_1 \quad (44)$$

all $X \in \mathcal{C}$, and it follows that \mathcal{C} is bounded in \mathcal{L}^1 . Thus to establish the statement of the proposition it suffices to show that if $R_\infty < 0$ then $\{\pi_t\}$ is not bounded in \mathcal{L}^1 . Keeping in mind that $\mathbb{E}[\pi_t] = P_{0t}$, we shall show that if $R_\infty < 0$, then for any choice of $\gamma > 0$ there exists a time t^* such that $P_{0t} > \gamma$ for all $t \geq t^*$. By virtue of (40), we have

$$P_{0t} = \exp \left[-R_\infty t + \frac{1}{k} (1 - e^{-kt}) (R_\infty - r_0) - \frac{1}{4} \frac{\sigma^2}{k^3} (1 - e^{-kt})^2 \right]. \quad (45)$$

It follows that

$$\log P_{0t} \geq -R_\infty t + \frac{1}{k} (R_\infty - r_0) \mathbb{1}\{R_\infty - r_0 \leq 0\} - \frac{1}{4} \frac{\sigma^2}{k^3}. \quad (46)$$

Therefore, let us define t^* by setting

$$-R_\infty t^* + \frac{1}{k} (R_\infty - r_0) \mathbb{1}\{R_\infty - r_0 \leq 0\} - \frac{1}{4} \frac{\sigma^2}{k^3} = \log \gamma, \quad (47)$$

or equivalently

$$t^* = \frac{1}{R_\infty} \left(\frac{1}{k} (R_\infty - r_0) \mathbb{1}\{R_\infty - r_0 \leq 0\} - \frac{1}{4} \frac{\sigma^2}{k^3} - \log \gamma \right). \quad (48)$$

Then it follows as a consequence that if $R_\infty < 0$ then for all $t > t^*$ we have $P_{0t} > \gamma$. This shows that if $R_\infty < 0$ then the pricing kernel is not UI. \square

Proposition 4 *If $R_\infty = 0$ then $\{\pi_t\}$ is not uniformly integrable.*

Proof. The situation when $R_\infty = 0$ is more delicate. The pricing kernel fails the \mathcal{L}^p test if $R_\infty = 0$, so we cannot conclude that it is UI. On the other hand, the pricing kernel is bounded in \mathcal{L}^1 if $R_\infty = 0$, so we cannot conclude that it is *not* UI. Thus when $R_\infty = 0$ the simple tests give us no information and we need to look at the definition of uniform integrability and ask whether (36) holds. We shall demonstrate that if $R_\infty = 0$ then (36) does not hold, and therefore the pricing kernel is not UI. First, let us define

$$\alpha_{st} = \frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(s-t)}. \quad (49)$$

Using (19) and (25) we can write

$$\pi_t = P_{0t} \exp \left(\int_0^t \alpha_{st} dW_s - \frac{1}{2} \int_0^t \alpha_{st}^2 ds \right). \quad (50)$$

Thus, for each value of t the pricing kernel is of the form

$$\pi_t = P_{0t} \exp \left(A_t Z - \frac{1}{2} A_t^2 \right), \quad (51)$$

where Z is normally distributed with mean zero and variance unity, and where we define A_t (which we take to be positive) by

$$A_t^2 = \int_0^t \alpha_{st}^2 ds. \quad (52)$$

It follows that

$$\mathbb{E}[\pi_t \mathbb{1}\{\pi_t > \delta\}] = P_{0t} \mathbb{E} \left[\exp \left(A_t Z - \frac{1}{2} A_t^2 \right) \mathbb{1} \left\{ Z > \frac{\log \delta - \log P_{0t} + \frac{1}{2} A_t^2}{A_t} \right\} \right]. \quad (53)$$

The expectation can be computed by standard techniques, leading to the following formula:

$$\mathbb{E}[\pi_t \mathbb{1}\{\pi_t > \delta\}] = P_{0t} N \left(\frac{\log P_{0t} + \frac{1}{2} A_t^2 - \log \delta}{A_t} \right). \quad (54)$$

Recall from (45) with $R_\infty = 0$ that

$$\log P_{0t} = -r_0 \frac{1}{k} (1 - e^{-kt}) - \frac{1}{4} \frac{\sigma^2}{k^3} (1 - e^{-kt})^2, \quad (55)$$

which is bounded. We thus have

$$\begin{aligned} \sup_t \mathbb{E}[\pi_t \mathbb{1}(\pi_t > \delta)] &\geq \exp \left[\inf_u \log P_{0u} \right] \sup_t N \left(\frac{\log P_{0t} + \frac{1}{2} A_t^2 - \log \delta}{A_t} \right) \\ &\geq \exp \left[\inf_u \log P_{0u} \right] \sup_t N \left[\frac{\inf_u (\log P_{0u}) + \frac{1}{2} A_t^2 - \log \delta}{A_t} \right]. \end{aligned} \quad (56)$$

It follows from

$$\inf_t \log P_{0t} = -\frac{r_0}{k} \mathbb{1}(r_0 > 0) - \frac{1}{4} \frac{\sigma^2}{k^3} \quad (57)$$

that

$$\begin{aligned} \sup_t \mathbb{E}[\pi_t \mathbb{1}(\pi_t > \delta)] &\geq \exp \left[-\frac{r_0}{k} \mathbb{1}(r_0 > 0) - \frac{1}{4} \frac{\sigma^2}{k^3} \right] \\ &\quad \times \sup_t N \left[\frac{1}{A_t} \left(-\frac{r_0}{k} \mathbb{1}(r_0 > 0) - \frac{1}{4} \frac{\sigma^2}{k^3} + \frac{1}{2} A_t^2 - \log \delta \right) \right]. \end{aligned} \quad (58)$$

Since $\lim_{t \rightarrow \infty} A_t = \infty$, the supremum on the right side is achieved in the limit as t approaches infinity. As a consequence, we have

$$\sup_t \mathbb{E}[\pi_t \mathbb{1}(\pi_t > \delta)] \geq \exp \left[-\frac{r_0}{k} \mathbb{1}(r_0 > 0) - \frac{1}{4} \frac{\sigma^2}{k^3} \right], \quad (59)$$

which implies that

$$\lim_{\delta \rightarrow \infty} \sup_t \mathbb{E}[\pi_t \mathbb{1}(\pi_t > \delta)] > 0, \quad (60)$$

and hence that the pricing kernel is not UI. \square

V. LONG-BOND RETURN PROCESS

The return at time t on an investment of one unit of account in the long bond is defined by the expression

$$L_t = \lim_{T \rightarrow \infty} \frac{P_{tT}}{P_{0T}}, \quad (61)$$

providing the limit exists in a suitable sense (Flesaker & Hughston 1996). We refer to $\{L_t\}_{t \geq 0}$ as the long-bond return process. In the following, we consider the long-bond return process in the Vasicek model. We shall show that the relevant limit exists and that it can be worked out explicitly. Using the formula for the discount bond price, we find that

$$\begin{aligned} \log \frac{P_{tT}}{P_{0T}} &= R_\infty t + \frac{1}{k}(r_0 - r_t) - \frac{1}{k}(R_\infty - r_t)e^{-k(T-t)} + \frac{1}{k}(R_\infty - r_0)e^{-kT} \\ &\quad + \frac{\sigma^2}{4k^3} \left((1 - e^{-kT})^2 - (1 - e^{-k(T-t)})^2 \right). \end{aligned} \quad (62)$$

One sees that the limit of this expression for large T is given by

$$\lim_{T \rightarrow \infty} \log \frac{P_{tT}}{P_{0T}} = R_\infty t + \frac{1}{k}(r_0 - r_t). \quad (63)$$

It follows that the long-bond return process is

$$L_t = \exp \left[R_\infty t + \frac{1}{k}(r_0 - r_t) \right]. \quad (64)$$

If we recall expression (16) for the pricing kernel in the Vasicek model, and expression (31) for the asymptotic rate, we deduce that the product of the pricing kernel and the long-bond return is given by

$$\pi_t L_t = \exp \left[\left(\frac{\sigma}{k} - \lambda \right) W_t - \frac{1}{2} \left(\frac{\sigma}{k} - \lambda \right)^2 t \right]. \quad (65)$$

This shows that the return on a unit investment in the long bond takes the form of a geometric Brownian motion asset in the Vasicek model, with volatility σ/k and with a Vasicek-type integrated interest rate. More specifically, we have

$$L_t = \exp \left[\int_0^t \left(r_s + \frac{\lambda \sigma}{k} \right) ds + \frac{\sigma}{k} W_t - \frac{1}{2} \left(\frac{\sigma}{k} \right)^2 t \right]. \quad (66)$$

The significance of the martingale $\{M_t\}_{t \geq 0}$ defined by $M_t = \pi_t L_t$ is that it acts as the change-of-measure density from the physical measure \mathbb{P} to the so-called terminal measure (or long forward measure) introduced in Flesaker & Hughston (1996). To see this, recall that to change from \mathbb{P} to the measure associated with a given numeraire $\{N_t\}$, the change-of-measure martingale is given by $\{\pi_t N_t\}$. For example, to change from \mathbb{P} to the risk-neutral measure associated with the use of the money market account as numeraire, the change-of-measure martingale is $\{\pi_t B_t\}$. In the present context, the terminal measure agrees with \mathbb{P} in the Vasicek model if and only if $M_t = 1$ for $t \geq 0$, which holds if and only if

$$\lambda = \frac{\sigma}{k}. \quad (67)$$

The condition that the terminal measure and the physical measure agree has been shown in Qin & Linetsky (2014) to be equivalent to the assumptions made in the recovery theorem of Ross (2015). Thus, we see that in the Vasicek model under the Ross recovery assumption one can infer the market price of risk from the current price levels of options on discount bonds, since such option prices depend on the ratio σ/k . However, there is no *a priori* reason to believe that the interest-rate market price of risk should be equal in general to the volatility of the long-bond return process. In fact, we have the following:

Proposition 5 *For any arbitrage-free interest-rate model based on a Brownian filtration, Ross recovery holds if and only if the interest-rate market price of risk is equal to the volatility of the long-bond return process.*

Proof. We know that Ross recovery holds if and only if the terminal measure coincides with the physical measure, or equivalently $M_t = 1$ for all $t \geq 0$, which holds if and only if $L_t = 1/\pi_t$ for all $t \geq 0$, which holds if and only if the interest-rate market price of risk agrees with the volatility of the return on the long bond. \square

This observation suggests that it is unlikely that Ross recovery will be observed in financial markets, a view supported by empirical evidence (Borovička *et al* 2014, Qin *et al* 2016).

VI. GEOMETRIC LÉVY MODELS

Are the foregoing conclusions—in particular, those addressing the feasibility of Ross recovery and those relating the positivity of the long rate of interest to the uniform integrability of the pricing kernel—specific to markets based on Brownian filtrations? To investigate this question, we consider the more general case of markets based on Lévy filtrations.

To begin, it will be useful if we briefly review the pricing kernel approach to geometric Lévy models. In such models, the pricing kernel method has the advantage that it brings to light the relations between risk, risk aversion, and return for models with price jumps (Brody *et al* 2012). Let $\{\xi_t\}$ be a Lévy process, and $\lambda > 0$ the level of risk aversion. We assume that $\{\xi_t\}$ satisfies a moment condition of the form

$$\mathbb{E}[\exp \alpha \xi_t] < \infty \quad \text{for } t \geq 0 \text{ and } \alpha \in A, \quad (68)$$

for some connected set $A \subset \mathbb{R}$ containing the origin. The pricing kernel of a geometric Lévy model, with constant interest rate r , is given by

$$\pi_t = e^{-rt} e^{-\lambda \xi_t - t\psi(-\lambda)}. \quad (69)$$

Here $\{\psi(\alpha)\}_{\alpha \in A}$, is the so-called Lévy exponent, defined by

$$\mathbb{E}[e^{\alpha \xi_t}] = e^{\psi(\alpha)t}. \quad (70)$$

It is straightforward to check that the Lévy exponent is a convex function. Since the product of the pricing kernel and the price $\{S_t\}$ of a non-dividend-paying asset is a \mathbb{P} -martingale, we may suppose that there is a $\beta \in A$ such that

$$\pi_t S_t = S_0 e^{\beta \xi_t - t\psi(\beta)}. \quad (71)$$

Writing $\sigma = \beta + \lambda$, we thus deduce that

$$S_t = S_0 e^{rt + R(\lambda, \sigma)t + \sigma \xi_t - t\psi(\sigma)}, \quad (72)$$

where the excess rate of return $R(\lambda, \sigma)$ is given by

$$R(\lambda, \sigma) = \psi(\sigma) + \psi(-\lambda) - \psi(\sigma - \lambda). \quad (73)$$

A short calculation shows that $R(\lambda, \sigma)$ is bilinear in λ and σ if and only if $\{\xi_t\}$ is a Brownian motion (Brody *et al* 2012). It follows that the widely popularized interpretation of λ as a “market price of risk”, which is valid for models based on a Brownian filtration, does not carry through to the general Lévy regime. Nevertheless, the notion of excess rate of return is well defined, and the strict convexity of the Lévy exponent implies that the excess rate of return is both an increasing function of λ and an increasing function of σ .

It is worth drawing attention to the fact that the value of the asset given by (72) does not depend on the drift of the Lévy process. Therefore without loss of generality we can set the drift of the Lévy process equal to zero. In that case we refer to $\{\xi_t\}$ as a compensated Lévy process. This implies that $\mathbb{E}[\xi_t] = 0$ and that $\{\xi_t\}$ is a martingale. For example, if $\{N_t\}$ is the standard Poisson process, with jump rate μ , then the associated compensated Lévy process is given by $\xi_t = N_t - \mu t$. With these conventions in mind we proceed to establish the following lemmas, which will be useful in what follows.

Lemma 1 *Let $\{\psi(\alpha)\}_{\alpha \in A}$ be the Lévy exponent of a compensated Lévy process $\{\xi_t\}$ that admits exponential moments for some connected set $A \subset \mathbb{R}$ containing the origin. Then ψ is strictly positive on its domain of definition, except at the origin, where it vanishes.*

Proof. Differentiating each side of (70) and setting $\alpha = 0$, we obtain $\mathbb{E}[\xi_t] = \psi'(0)t$ for all $t \geq 0$. Since $\{\psi(\alpha)\}$ is by assumption a compensated Lévy process, it follows that $\psi'(0) = 0$. Hence, the curve $\psi : A \rightarrow \mathbb{R}$ defined by $\alpha \in A \rightarrow \psi(\alpha)$ has a horizontal tangent at the origin. Since ψ is strictly convex, and thus lies above any of its tangents except at the point where the tangent touches the curve, we conclude that ψ is strictly positive except at the origin. At the origin, we have $\psi(0) = 0$, which follows from the definition of the Lévy exponent. \square

Lemma 2 *Let $A \subset \mathbb{R}$ be a connected set containing the origin, and let $f : A \rightarrow \mathbb{R}$ be a nonnegative strictly convex function, differentiable on A and vanishing at 0. Then it holds that $xf'(x) > f(x)$ for all $x \in A$ except at $x = 0$.*

Proof. Let $x \in A$. If $x > 0$, then by the mean value theorem there exists $y \in (0, x)$ such that $f(x) = xf'(y)$. Since f takes its minimum at the origin and is strictly convex, it follows that $f'(y) < f'(x)$. Therefore, $f(x) < xf'(x)$, as required. On the other hand, if $x < 0$, then the mean value theorem says that there exists $y \in (x, 0)$ such that $f(x) = xf'(y)$. But since f is strictly convex with a minimum at the origin, it follows that $0 < f'(x) < f'(y)$, and thus $xf'(y) < xf'(x)$, since $x < 0$. Therefore, $f(x) < xf'(x)$. \square

VII. CONSTRUCTION OF LÉVY-VASICEK MODEL

Our objective going forward is to investigate properties of the pricing kernel in the Lévy analogue of the Vasicek model. Specifically, we are interested in an Ornstein-Uhlenbeck-type short rate model driven by a Lévy process. Such models have been investigated, for example, in Norberg (2004). Remarkably, the condition $\lambda = \sigma/k$ for Ross recovery in the Vasicek model is unchanged in its Lévy-Vasicek counterpart. We shall also show that the pricing kernel is UI if and only if the long rate of interest is strictly positive.

The pricing kernel method allows us to work out the details of the general Lévy-Vasicek model in the \mathbb{P} -measure. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and introduce a one-dimensional Lévy process $\{\xi_t\}_{t \geq 0}$. We assume that (68) holds for some connected set $A \subset \mathbb{R}$ containing the origin, and we write $\psi(\alpha)$ for the Lévy exponent, defined for $\alpha \in A$. The pricing kernel in the Lévy-Vasicek model then takes the form

$$\pi_t = \exp \left[- \int_0^t r_s ds - \lambda \xi_t - \psi(-\lambda) t \right], \quad (74)$$

where the short rate is a Lévy-Ornstein-Uhlenbeck process satisfying a dynamical equation of the form

$$dr_t = k(\theta - r_t) dt - \sigma d\xi_t. \quad (75)$$

The parameters of the model here have essentially the same interpretation as those of the classical Vasicek model. The only difference is that in the Lévy case we let the volatility parameter have dimensions of inverse time. The Lévy process itself is taken to be dimensionless. This is of course quite natural in the case of counting processes. In the situation where $\{\xi_t\}$ is a Brownian motion, it will be understood that a standard Brownian motion is multiplied by an appropriate volatility parameter to make the overall process dimensionless. Without loss of generality we can set the drift of the Lévy process equal to zero by absorbing any drift into the definition of the mean-reversion level. Thus in what follows we assume that $\{\xi_t\}$ is a compensated Lévy process. As in the Brownian case, we find by use of an integrating factor that the short rate is given by

$$r_t = \theta + (r_0 - \theta)e^{-kt} - \sigma \int_0^t e^{k(s-t)} d\xi_s. \quad (76)$$

The integrated short rate can be worked out by a calculation that parallels that of the classical Vasicek model, with the following result:

$$\int_0^t r_s ds = \theta t + \frac{1}{k}(r_0 - r_t) - \frac{\sigma}{k} \xi_t. \quad (77)$$

It follows that the pricing kernel in the Lévy-Vasicek model can be written in the form

$$\pi_t = \exp \left[-(\theta + \psi(-\lambda))t + \left(\frac{\sigma}{k} - \lambda \right) \xi_t - \frac{1}{k}(r_0 - r_t) \right]. \quad (78)$$

Alternatively, if we insert the expression for r_t given in (76) then we obtain

$$\pi_t = \exp \left[-(\theta + \psi(-\lambda))t - \frac{1}{k}(1 - e^{-kt})(r_0 - \theta) + \int_0^t \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-t)} \right) d\xi_u \right]. \quad (79)$$

VIII. DISCOUNT BONDS IN LÉVY-VASICEK MODEL

We proceed to obtain an expression for the price of a discount bond. By (79) we have

$$\log \pi_T = -(\theta + \psi(-\lambda))t - \frac{1}{k}(1 - e^{-kT})(r_0 - \theta) + \int_0^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right) d\xi_u, \quad (80)$$

from which we deduce that

$$\begin{aligned} \log \pi_T &= -(\theta + \psi(-\lambda))T - \frac{1}{k}(1 - e^{-kT})(r_0 - \theta) + \int_0^t \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right) d\xi_u \\ &\quad + \int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right) d\xi_u. \end{aligned} \quad (81)$$

Hence,

$$\begin{aligned} \mathbb{E}_t[\pi_T] &= \exp \left[-(\theta + \psi(-\lambda))T - \frac{1}{k}(1 - e^{-kT})(r_0 - \theta) + \int_0^t \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right) d\xi_u \right] \\ &\quad \times \mathbb{E}_t \exp \left[\int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right) d\xi_u \right]. \end{aligned} \quad (82)$$

To work out the conditional expectation on the right side we use the identity

$$\mathbb{E}_t \exp \left[\int_t^T \alpha_s d\xi_s \right] = \exp \int_t^T \psi(\alpha_s) ds, \quad (83)$$

valid for any Borel function $\{\alpha_s\}_{s \geq 0}$ taking values in the interval A . It follows that

$$\mathbb{E}_t \exp \left[\int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right) d\xi_u \right] = \exp \left[\int_t^T \psi \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right) du \right]. \quad (84)$$

Therefore, by expression (5) for the discount bond we obtain

$$\begin{aligned} \log P_{tT} &= -(\theta + \psi(-\lambda))(T - t) - \frac{1}{k}(e^{-kt} - e^{-kT})(r_0 - \theta) \\ &\quad + \int_t^T \psi \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-T)} \right) du + \frac{\sigma}{k}(1 - e^{k(t-T)}) \int_0^t e^{k(u-t)} d\xi_u. \end{aligned} \quad (85)$$

As a consequence of (76) we can write

$$\sigma \int_0^t e^{k(u-t)} d\xi_u = \theta + (r_0 - \theta)e^{-kt} - r_t. \quad (86)$$

Thus, the terms involving $r_0 - \theta$ in (85) cancel, and we are left with the following expression for the price of a T -maturity discount bond:

$$P_{tT} = \exp \left[-(\theta + \psi(\lambda))(T - t) + \int_t^T \psi(\alpha_{uT}) du + \frac{1}{k}(1 - e^{k(t-T)})(\theta - r_t) \right], \quad (87)$$

where for $0 \leq s \leq t$ we set

$$\alpha_{ut} = \frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(u-t)}. \quad (88)$$

To investigate the asymptotic bond yield, or exponential long rate, first we show that

$$\lim_{T \rightarrow \infty} \frac{1}{T-t} \int_t^T \psi(\alpha_{sT}) ds = \psi\left(\frac{\sigma}{k} - \lambda\right). \quad (89)$$

We note that the derivative of the numerator with respect to T is given by

$$\begin{aligned} \frac{d}{dT} \int_t^T \psi(\alpha_{sT}) ds &= \psi(\alpha_{TT}) + \sigma \int_t^T \psi'(\alpha_{sT}) e^{k(s-T)} ds \\ &= \psi(\alpha_{TT}) - \int_t^T \frac{d}{ds} \psi(\alpha_{sT}) ds \\ &= \psi(\alpha_{tT}). \end{aligned} \quad (90)$$

Thus, applying l'Hospital's rule we obtain

$$\lim_{T \rightarrow \infty} \frac{1}{T-t} \int_t^T \psi(\alpha_{sT}) ds = \lim_{T \rightarrow \infty} \psi(\alpha_{tT}) = \psi\left(\frac{\sigma}{k} - \lambda\right), \quad (91)$$

establishing (89). One sees that in the Lévy-Vasicek model, as in the Brownian case, the long rate does not depend on t , and we have the following:

$$R_\infty = - \lim_{T \rightarrow \infty} \frac{1}{T-t} \log P_{tT} = \theta + \psi(-\lambda) - \psi\left(\frac{\sigma}{k} - \lambda\right). \quad (92)$$

IX. UNIFORM INTEGRABILITY IN LÉVY-VASICEK MODEL

A natural question that emerges, in view of our conjecture that an economically admissible interest rate model should be accompanied by a UI pricing kernel, is whether and for what choice of parameters the pricing kernel in a Lévy-Vasicek model is UI. For this purpose, it will be useful to express the pricing kernel in the form

$$\pi_t = \exp\left(-(\theta + \psi(-\lambda))t - \frac{1}{k}(r_0 - \theta)(1 - e^{-kt}) + \int_0^t \alpha_{st} d\xi_s\right), \quad (93)$$

or equivalently

$$\pi_t = \exp\left(-R_\infty t - \psi\left(\frac{\sigma}{k} - \lambda\right)t - \frac{1}{k}(r_0 - \theta)(1 - e^{-kt}) + \int_0^t \alpha_{st} d\xi_s\right), \quad (94)$$

where we recall the definition (88) for α_{st} . It follows that

$$\log \mathbb{E}[\pi_t] = -R_\infty t - \frac{1}{k}(r_0 - \theta)(1 - e^{-kt}) + \int_0^t \psi(\alpha_{st}) ds - \psi\left(\frac{\sigma}{k} - \lambda\right)t. \quad (95)$$

In the asymptotic considerations that follow, we adopt the following conventions. Given a pair of functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ and $g : \mathbb{R}^+ \rightarrow \mathbb{R} \setminus \{0\}$, we say that f is $O(g)$ for large t if

$$\limsup_{t \rightarrow \infty} \left| \frac{f(t)}{g(t)} \right| < \infty, \quad (96)$$

and we say that f is $o(g)$ for large t if

$$\lim_{t \rightarrow \infty} \frac{f(t)}{g(t)} = 0. \quad (97)$$

With reference to the integral appearing on the right side of (95), we shall show that

$$\int_0^t \psi(\alpha_{st}) \, ds = \psi\left(\frac{\sigma}{k} - \lambda\right) t + O(1) \quad (98)$$

for large t . We note that

$$\left| \int_0^t \left(\psi(\alpha_{st}) - \psi\left(\frac{\sigma}{k} - \lambda\right) \right) \, ds \right| \leq \int_0^t \left| \psi(\alpha_{st}) - \psi\left(\frac{\sigma}{k} - \lambda\right) \right| \, ds. \quad (99)$$

But it follows from the mean value theorem that, for a fixed value of s , there exists a ρ in the open interval $(-\sigma k^{-1} e^{k(s-t)}, 0)$ such that

$$\psi\left(\frac{\sigma}{k} - \lambda\right) = \psi(\alpha_{st}) + \frac{\sigma}{k} e^{k(s-t)} \psi'\left(\frac{\sigma}{k} - \lambda + \rho\right). \quad (100)$$

Hence,

$$\left| \psi(\alpha_{st}) - \psi\left(\frac{\sigma}{k} - \lambda\right) \right| = \frac{\sigma}{k} e^{k(s-t)} \left| \psi'\left(\frac{\sigma}{k} - \lambda + \rho\right) \right|. \quad (101)$$

Recall from Lemma 1 that ψ is a nonnegative strictly convex function taking its minimum value at zero. Thus, over the relevant range of ρ , the maximum of $|\psi'(\sigma/k - \lambda + \rho)|$ is taken, depending on the value of s , either at the left boundary $\rho = -(\sigma/k)e^{k(s-t)}$ or at the right boundary $\rho = 0$. More precisely, there exists a $t' \in [0, t]$ such that as s varies, the maximum is achieved at the right boundary whenever $s \in (0, t')$, and at the left boundary whenever $s \in (t', t)$. Thus, when $s \in (0, t')$ we have

$$\left| \psi(\alpha_{st}) - \psi\left(\frac{\sigma}{k} - \lambda\right) \right| \leq \frac{\sigma}{k} e^{k(s-t)} \left| \psi'\left(\frac{\sigma}{k} - \lambda\right) \right|, \quad (102)$$

and when $s \in (t', t)$ we have

$$\left| \psi(\alpha_{st}) - \psi\left(\frac{\sigma}{k} - \lambda\right) \right| \leq \frac{\sigma}{k} e^{k(s-t)} \left| \psi'\left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k} e^{k(s-t)}\right) \right| \leq \frac{\sigma}{k} e^{k(s-t)} |\psi'(-\lambda)|. \quad (103)$$

In the last step here we have made use of the fact that since ψ is convex, $|\psi'|$ is decreasing on the negative half line. Reverting to the right side of (99), we thus deduce that

$$\begin{aligned} \int_0^t \left| \psi(\alpha_{st}) - \psi\left(\frac{\sigma}{k} - \lambda\right) \right| \, ds &= \int_0^{t'} \left| \psi(\alpha_{st}) - \psi\left(\frac{\sigma}{k} - \lambda\right) \right| \, ds + \int_{t'}^t \left| \psi(\alpha_{st}) - \psi\left(\frac{\sigma}{k} - \lambda\right) \right| \, ds \\ &\leq \frac{\sigma}{k} \psi'\left(\frac{\sigma}{k} - \lambda\right) \int_0^{t'} e^{k(s-t)} \, ds + \frac{\sigma}{k} |\psi'(-\lambda)| \int_{t'}^t e^{k(s-t)} \, ds \\ &= \frac{\sigma}{k^2} \psi'\left(\frac{\sigma}{k} - \lambda\right) e^{-kt} (e^{kt'} - 1) + \frac{\sigma}{k^2} |\psi'(-\lambda)| (1 - e^{-k(t-t')}) \\ &\leq \frac{\sigma}{k^2} \psi'\left(\frac{\sigma}{k} - \lambda\right) + \frac{\sigma}{k^2} |\psi'(-\lambda)|, \end{aligned} \quad (104)$$

and hence

$$\left| \int_0^t \psi(\alpha_{st}) \, ds - \psi\left(\frac{\sigma}{k} - \lambda\right) t \right| \leq \frac{\sigma}{k^2} \psi'\left(\frac{\sigma}{k} - \lambda\right) + \frac{\sigma}{k^2} |\psi'(-\lambda)|. \quad (105)$$

It follows that

$$\limsup_{t \rightarrow \infty} \left| \int_0^t \psi(\alpha_{st}) \, ds - \psi\left(\frac{\sigma}{k} - \lambda\right) t \right| < \infty, \quad (106)$$

which establishes (98).

With these preparations at hand, we are ready to return to our considerations of the uniform integrability of the pricing kernel.

Proposition 6 *If $R_\infty > 0$ then $\{\pi_t\}$ is uniformly integrable.*

Proof. We shall use an \mathcal{L}^p test. Specifically, we show that if $R_\infty > 0$ then there exists a $p > 1$ such that $\sup_t \mathbb{E}[\pi_t^p] < \infty$. We claim that it suffices to prove that if $R_\infty > 0$ then $\lim_{t \rightarrow \infty} \mathbb{E}[\pi_t^p] = 0$ for some $p > 1$. To see this, note that if $\lim_{t \rightarrow \infty} \mathbb{E}[\pi_t^p] = 0$ holds for some $p > 1$, there exist positive constants C and T such that $\mathbb{E}[\pi_t^p] < C$ for all $t \geq T$. But then

$$\sup_t \mathbb{E}[\pi_t^p] \leq \max\left(C, \sup_{t \leq T} \mathbb{E}[\pi_t^p]\right). \quad (107)$$

Since continuous functions are bounded on compact intervals, we see that $\sup_{t \leq T} \mathbb{E}[\pi_t^p]$ is bounded, and thus $\sup_t \mathbb{E}[\pi_t^p] < \infty$, as required. Let us therefore show that $\lim_{t \rightarrow \infty} \mathbb{E}[\pi_t^p] = 0$ for some $p > 1$. It follows from (94) after a calculation that

$$\log \mathbb{E}[\pi_t^p] = -pR_\infty t - \frac{p}{k}(r_0 - \theta)(1 - e^{-kt}) + \int_0^t \psi(p\alpha_{st}) \, ds - p\psi\left(\frac{\sigma}{k} - \lambda\right) t, \quad (108)$$

and we can use (98) to see that

$$\begin{aligned} \log \mathbb{E}[\pi_t^p] &= -pR_\infty t - \frac{p}{k}(r_0 - \theta)(1 - e^{-kt}) \\ &\quad + \left(\psi\left(p\left(\frac{\sigma}{k} - \lambda\right)\right) - p\psi\left(\frac{\sigma}{k} - \lambda\right) \right) t + O(1) \end{aligned} \quad (109)$$

for large t . Since ψ is continuous at $\sigma/k - \lambda$, we observe that the quantity

$$\left| \left(\psi\left(p\left(\frac{\sigma}{k} - \lambda\right)\right) - p\psi\left(\frac{\sigma}{k} - \lambda\right) \right) \right| \quad (110)$$

can be made arbitrarily small as p approaches unity from above. Keeping in mind that $R_\infty > 0$, we thus see that there exists a $p > 1$ such that

$$-R_\infty + \left(\psi\left(p\left(\frac{\sigma}{k} - \lambda\right)\right) - p\psi\left(\frac{\sigma}{k} - \lambda\right) \right) < 0, \quad (111)$$

and hence $\lim_{t \rightarrow \infty} \mathbb{E}[\pi_t^p] = 0$, which is the required result. \square

Proposition 7 *If $R_\infty < 0$ then $\{\pi_t\}$ is not uniformly integrable.*

Proof. We shall show that $\{\pi_t\}$ is not bounded in \mathcal{L}^1 , which implies that $\{\pi_t\}$ is not UI. It follows from (95) and (98) that for large t one has

$$\log \mathbb{E}[\pi_t] = -R_\infty t - \frac{1}{k}(r_0 - \theta)(1 - e^{-kt}) + O(1). \quad (112)$$

If $R_\infty < 0$, we have

$$\lim_{t \rightarrow \infty} \mathbb{E}[\pi_t] = \infty. \quad (113)$$

Thus, the pricing kernel is not bounded in \mathcal{L}^1 . \square

Proposition 8 *If $R_\infty = 0$ then $\{\pi_t\}$ is not uniformly integrable.*

Proof. We recall that the pricing kernel is UI if and only if (36) holds. Using (87) and (93) we have

$$\pi_t = P_{0t} \exp \left(\int_0^t \alpha_{st} d\xi_s - \int_0^t \psi(\alpha_{st}) ds \right). \quad (114)$$

As a consequence, we see that

$$\mathbb{E}[\pi_t \mathbb{1}\{\pi_t > \delta\}] = P_{0t} \mathbb{E} \left[\exp \left(\int_0^t \alpha_{st} d\xi_s - \int_0^t \psi(\alpha_{st}) ds \right) \mathbb{1} \left\{ \int_0^t \alpha_{st} d\xi_s > B(t, \delta) \right\} \right], \quad (115)$$

where we define

$$B(t, \delta) = \log \delta - \log P_{0t} + \int_0^t \psi(\alpha_{st}) ds. \quad (116)$$

Equivalently, by (87) we have

$$B(t, \delta) = \log \delta + R_\infty t + \frac{1}{k}(r_0 - \theta)(1 - e^{-kt}) + \psi \left(\frac{\sigma}{k} - \lambda \right) t. \quad (117)$$

We introduce a new measure \mathbb{P}^* on \mathcal{F}_t by setting

$$\mathbb{P}^*(A) = \mathbb{E} \left[\exp \left(\int_0^t \alpha_{st} d\xi_s - \int_0^t \psi(\alpha_{st}) ds \right) \mathbb{1}\{A\} \right] \quad (118)$$

for $A \in \mathcal{F}_t$. Writing \mathbb{E}^* for expectation under \mathbb{P}^* we then have

$$\mathbb{E}[\pi_t \mathbb{1}\{\pi_t > \delta\}] = P_{0t} \mathbb{E}^* \left[\mathbb{1} \left\{ \int_0^t \alpha_{st} d\xi_s > B(t, \delta) \right\} \right]. \quad (119)$$

Let us introduce a positive constant ω with units of inverse time, making ωt dimensionless. Thus ω is a fixed ‘‘rate’’. To proceed, we need the following results regarding the mean and variance of the random variable

$$\int_0^t \alpha_{st} d\xi_s \quad (120)$$

under \mathbb{P}^* . If $R_\infty = 0$, then for large t we have

$$\mathbb{E}^* \left[\int_0^t \alpha_{st} d\xi_s \right] - B(t, \omega t) = Ct + o(t), \quad (121)$$

where

$$C = \left(\frac{\sigma}{k} - \lambda \right) \psi' \left(\frac{\sigma}{k} - \lambda \right) - \psi \left(\frac{\sigma}{k} - \lambda \right) \quad (122)$$

is a positive constant, and

$$\text{Var}^* \left[\int_0^t \alpha_{st} d\xi_s \right] = O(t). \quad (123)$$

To see (121), (122), and (123), note that for u sufficiently close to zero we have

$$\mathbb{E}^* \left[\exp \left(u \int_0^t \alpha_{st} d\xi_s \right) \right] = \exp \left(\int_0^t (\psi((1+u)\alpha_{st}) - \psi(\alpha_{st})) ds \right). \quad (124)$$

Taking a derivative with respect to u on both sides of this equation and setting $u = 0$ gives

$$\mathbb{E}^* \left[\int_0^t \alpha_{st} d\xi_s \right] = \int_0^t \psi'(\alpha_{st}) \alpha_{st} ds. \quad (125)$$

A calculation then shows that

$$\frac{d}{dt} \mathbb{E}^* \left[\int_0^t \alpha_{st} d\xi_s \right] = \alpha_{0t} \psi'(\alpha_{0t}), \quad (126)$$

and thus, by l'Hospital's rule,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}^* \left[\int_0^t \alpha_{st} d\xi_s \right] = \lim_{t \rightarrow \infty} \alpha_{0t} \psi'(\alpha_{0t}) = \left(\frac{\sigma}{k} - \lambda \right) \psi' \left(\frac{\sigma}{k} - \lambda \right). \quad (127)$$

It follows that

$$\mathbb{E}^* \left[\int_0^t \alpha_{st} d\xi_s \right] = \left(\frac{\sigma}{k} - \lambda \right) \psi' \left(\frac{\sigma}{k} - \lambda \right) t + o(t) \quad (128)$$

for large t . Recall from (117) that $B(t, \delta)$ grows to leading order like $\psi(\sigma/k - \lambda)t$ when $R_\infty = 0$. This remains the case if we replace δ by ωt , since the growth of $\log \omega t$ is slower than linear growth. That is,

$$\mathbb{E}^* \left[\int_0^t \alpha_{st} d\xi_s \right] - B(t, \omega t) = \left[\left(\frac{\sigma}{k} - \lambda \right) \psi' \left(\frac{\sigma}{k} - \lambda \right) - \psi \left(\frac{\sigma}{k} - \lambda \right) \right] t + o(t) \quad (129)$$

for large t . The positivity of the coefficient of t on the right side follows from Lemma 2. We have thus arrived at equations (121) and (122). It remains to show equation (123). Taking the second derivative of (124) with respect to u and setting $u = 0$, we find

$$\mathbb{E}^* \left[\left(\int_0^t \alpha_{st} d\xi_s \right)^2 \right] = \left(\int_0^t \psi'(\alpha_{st}) \alpha_{st} ds \right)^2 + \int_0^t \psi''(\alpha_{st}) \alpha_{st}^2 ds. \quad (130)$$

Using (126), we then obtain

$$\begin{aligned} \text{Var}^* \left[\int_0^t \alpha_{st} d\xi_s \right] &= \mathbb{E}^* \left[\left(\int_0^t \alpha_{st} d\xi_s \right)^2 \right] - \left(\mathbb{E}^* \left[\int_0^t \alpha_{st} d\xi_s \right] \right)^2 \\ &= \int_0^t \psi''(\alpha_{st}) \alpha_{st}^2 ds. \end{aligned} \quad (131)$$

The limit

$$\lim_{t \rightarrow \infty} \frac{1}{t} \text{Var}^* \left[\int_0^t \alpha_{st} d\xi_s \right] = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \psi''(\alpha_{st}) \alpha_{st}^2 ds \quad (132)$$

is finite, which can be seen using l'Hospital's rule. Hence, (123) follows. Now let us define

$$c(t) = \mathbb{E}^* \left[\int_0^t \alpha_{st} d\xi_s \right] - B(t, \omega t), \quad (133)$$

and recall from (121) that $c(t)$ grows linearly for large t . Let t be large enough so that $c(t) > 0$. We recall that if X is a random variable such that $\text{Var}[X] < \infty$, then for any constant $c > 0$ we have the Chebyshev inequality

$$\mathbb{P} [|X - \mathbb{E}[X]| \geq c] \leq \frac{1}{c^2} \text{Var}[X]. \quad (134)$$

In the present context it follows that

$$\mathbb{P}^* \left[\left| \int_0^t \alpha_{st} d\xi_s - \mathbb{E}^* \left[\int_0^t \alpha_{st} d\xi_s \right] \right| \geq c(t) \right] \leq \frac{1}{c(t)^2} \text{Var}^* \left[\int_0^t \alpha_{st} d\xi_s \right]. \quad (135)$$

Since both $c(t)$ and the variance grow linearly in t by (121)–(123), we see that

$$\lim_{t \rightarrow \infty} \mathbb{P}^* \left[\left| \int_0^t \alpha_{st} d\xi_s - \mathbb{E}^* \left[\int_0^t \alpha_{st} d\xi_s \right] \right| \geq c(t) \right] = 0. \quad (136)$$

For t large enough so that $c(t) > 0$, we have

$$\begin{aligned} \mathbb{E}^* \left[\mathbf{1} \left\{ \int_0^t \alpha_{st} d\xi_s > B(t, \omega t) \right\} \right] &= \mathbb{P}^* \left[\int_0^t \alpha_{st} d\xi_s > B(t, \omega t) \right] \\ &\geq \mathbb{P}^* \left[\left| \int_0^t \alpha_{st} d\xi_s - \mathbb{E}^* \left[\int_0^t \alpha_{st} d\xi_s \right] \right| < c(t) \right] \\ &= 1 - \mathbb{P}^* \left[\left| \int_0^t \alpha_{st} d\xi_s - \mathbb{E}^* \left[\int_0^t \alpha_{st} d\xi_s \right] \right| \geq c(t) \right]. \end{aligned} \quad (137)$$

Taking the limit on both sides as t gets large, and using equation (136), one finds that

$$\lim_{t \rightarrow \infty} \mathbb{E}^* \left[\mathbf{1} \left\{ \int_0^t \alpha_{st} d\xi_s > B(t, \omega t) \right\} \right] = 1. \quad (138)$$

To proceed, recall from (36) and (119) that in order to show that the pricing kernel is not UI when $R_\infty = 0$ one needs to prove that

$$\limsup_{\delta \rightarrow \infty} \sup_t \left(P_{0t} \mathbb{E}^* \left[\mathbf{1} \left\{ \int_0^t \alpha_{st} d\xi_s > B(t, \delta) \right\} \right] \right) > 0. \quad (139)$$

To see that (139) holds, note that

$$\begin{aligned} & \limsup_{\delta \rightarrow \infty} \sup_t \left(P_{0t} \mathbb{E}^* \left[\mathbf{1} \left\{ \int_0^t \alpha_{st} d\xi_s > B(t, \delta) \right\} \right] \right) \\ &= \limsup_{T \rightarrow \infty} \sup_t \left(P_{0t} \mathbb{E}^* \left[\mathbf{1} \left\{ \int_0^t \alpha_{st} d\xi_s > B(t, \omega T) \right\} \right] \right) \\ &= \limsup_{T \rightarrow \infty} \sup_t \left(P_{0t} \mathbb{E}^* \left[\mathbf{1} \left\{ \int_0^t \alpha_{st} d\xi_s > B(t, \omega T) \right\} \right] \right) \\ &\geq \limsup_{T \rightarrow \infty} P_{0T} \mathbb{E}^* \left[\mathbf{1} \left\{ \int_0^T \alpha_{sT} d\xi_s > B(T, \omega T) \right\} \right] \\ &= \limsup_{T \rightarrow \infty} P_{0T}, \end{aligned} \quad (140)$$

where the final equality follows as a consequence of equation (138). Keeping in mind that for the case under consideration we have $R_\infty = 0$, it follows from (95) and (98) that P_{0T} is of the form

$$P_{0T} = \exp \left(-\frac{1}{k}(r_0 - \theta) (1 - e^{-kT}) + f(T) \right) \quad (141)$$

for some function $f(T)$ that satisfies

$$\limsup_{T \rightarrow \infty} |f(T)| < \infty. \quad (142)$$

We deduce that

$$\limsup_{T \rightarrow \infty} P_{0T} = \exp \left(-\frac{1}{k}(r_0 - \theta) \right) \limsup_{T \rightarrow \infty} \exp(f(T)) > 0. \quad (143)$$

Therefore, the right hand side of (140) is strictly positive, and thus we have shown that (139) holds, which concludes the proof. \square

X. LONG-BOND RETURN IN LÉVY-VASICEK MODEL

We proceed to determine the return on a unit investment in the long bond in a Lévy-Vasicek model. Note that

$$\lim_{T \rightarrow \infty} \log \frac{P_{tT}}{P_{0T}} = \lim_{T \rightarrow \infty} [-R_{tT}(T-t) + R_{0T}T], \quad (144)$$

from which it follows that

$$\begin{aligned} L_t &= (\theta + \psi(-\lambda))t + \frac{1}{k}(r_0 - r_t) - \lim_{T \rightarrow \infty} \int_0^t \psi(\alpha_{sT}) ds \\ &= (\theta + \psi(-\lambda))t + \frac{1}{k}(r_0 - r_t) - \psi \left(\frac{\sigma}{k} - \lambda \right) t \\ &= R_\infty t + \frac{1}{k}(r_0 - r_t). \end{aligned} \quad (145)$$

It is natural to ask whether it is possible to bring the long-bond return process into the form of a geometric price process. If we take into account relation (77) for the integrated short rate and definition (73) for the excess rate of return, then after some algebra we deduce that

$$L_t = \exp \left[\int_0^t \left(r_s + R \left(\lambda, \frac{\sigma}{k} \right) \right) ds + \frac{\sigma}{k} \xi_t - \psi \left(\frac{\sigma}{k} \right) t \right]. \quad (146)$$

Thus, the form of the long-bond return is indeed that of a geometric asset price, with a Lévy-Vasicek short rate r_t , risk aversion λ , and volatility σ/k . If we multiply this expression by the pricing kernel, then after some cancelation we obtain the geometric Lévy martingale

$$M_t = \exp \left[\left(\frac{\sigma}{k} - \lambda \right) \xi_t - \psi \left(\frac{\sigma}{k} - \lambda \right) t \right]. \quad (147)$$

This gives the result that Ross recovery holds in a Lévy-Vasicek model if and only if

$$\lambda = \frac{\sigma}{k}, \quad (148)$$

just as in the Brownian context. While it seems unlikely in practice that the interest-rate market price of risk is equal to the long-bond volatility, it would nevertheless be interesting to extend the analysis of Carr & Yu (2012), Borovička *et al* (2014), Qin *et al* (2016), and others to the context of Lévy models such as the ones developed here. It remains an open question whether a version of Ross recovery may yet survive if one works entirely in real terms. What is clear, however, is that the matters investigated in the present paper, in particular the uniform integrability of pricing kernels and the nature of the long-bond return process, are important areas of investigation, and that their examination in varied contexts is likely to offer further insights into the behaviour of financial models over long time horizons.

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- [1] Borovička, J., Hansen, L. P. & Scheinkman, A. J. (2014) Misspecified recovery. arXiv:1412.0042.
 - [2] Brody, D. C., Hughston, L. P. & Mackie, E. (2012) General theory of geometric Lévy models for dynamic asset pricing. *Proceedings of the Royal Society of London A* **468**, 1778-1798.
 - [3] Brody, D. C. & Hughston, L. P. (2016) Social discounting and the long rate of interest. *Mathematical Finance*. doi 10.1111/mafi.12122 (1-29).

- [4] Cairns, A. J. G. (1999) An alternative derivation of the Vasicek model. Herriot-Watt University, Department of Actuarial Mathematics and Statistics, technical note 99/13.
- [5] Carr, P. & Yu, J. (2012) Risk, return, and Ross recovery. *J. Derivatives* **20**, 38-59.
- [6] Cochrane, J. H. (2005) *Asset Pricing* (Princeton: Princeton University Press).
- [7] Cox, S. H. & Martin, J. D. (1983) Abandonment value and capital budgeting under uncertainty. *J. Econom. Bus.* **35**, 331-341.
- [8] Dothan, U. & Williams, J. T. (1978) Valuation of assets on a Markov state space. *Econom. Lett.* **1**, 163-166.
- [9] Duffie, D. (1992) *Dynamic Asset Pricing* (Princeton: Princeton University Press).
- [10] Dybvig, P. H., Ingersoll, J. E. & Ross, S. A. (1996) Long forward and zero-coupon rates can never fall. *Journal of Business* **69**, 1-25.
- [11] Eberlein, E. & Raible, S. (1999) Term structure models driven by general Lévy processes. *Mathematical Finance* **9**, 31-53.
- [12] Flesaker, B. & Hughston, L. P. (1996) Positive interest. *Risk* **9**, 46-49. Reprinted in *Vasicek and Beyond* (L. P. Hughston, ed.) London: Risk Publications (1996).
- [13] Flesaker, B. & Hughston, L. P. (1997) International models for interest rates and foreign exchange. *Net Exposure*, Issue 3, 55-79. Reprinted in *The New Interest Rate Models* (L. P. Hughston, ed.) London: Risk Publications (2000).
- [14] Garman, M. B. (1976) A general theory of asset valuation under diffusion state processes. Working Paper No. 50, Institute of Business and Economic Research, U. California, Berkeley.
- [15] Goldammer, V. & Schmock, U. (2012) Generalization of the Dybvig-Ingersoll-Ross theorem and asymptotic minimality. *Mathematical Finance* **22**, 185-213.
- [16] Hubalek, F., Klein, I. & Teichmann, J. (2002) A general proof of the Dybvig-Ingersoll-Ross theorem: long forward rates can never fall. *Mathematical Finance* **12**, 447-451.
- [17] Hunt, P. J. & Kennedy, J. E. (2004) *Financial Derivatives in Theory and Practice*, revised edition (Chichester: Wiley).
- [18] Kardaras, C. & Platen, E. (2012) On the Dybvig-Ingersoll-Ross theorem. *Mathematical Finance* **22**, 729-740.
- [19] Meyer, P. A (1996) *Probability and Potentials* (Waltham, MA: Blaisdell Publishing Company).
- [20] Norberg, R. (2004) Vasicek beyond the normal. *Mathematical Finance* **14**, 585-604.
- [21] Qin, L. & Linetsky, V. (2014) Long term risk: a martingale approach. arXiv:1411.3078
- [22] Qin, L., Linetsky, V. & Nie, N. (2016) Long forward probabilities, recovery and the term structure of bond risk premiums. arXiv:1601.06477
- [23] Rogers, L. C. G. (1997) The potential approach to the term structure of interest rates and foreign exchange rates. *Mathematical Finance* **7**, 157-176.
- [24] Ross, S. (1978) The current status of the capital asset pricing model. *J. Finance* **33**, 885-901.
- [25] Ross, S. (2015) The recovery theorem. *J. Finance* **70**, 615-648.
- [26] Rutkowski, M. (1997) A note on the Flesaker-Hughston model of the term structure of interest rates. *Applied Mathematical Finance* **4**, 151-163.
- [27] Vasicek, O. (1977) An equilibrium characterization of the term structure. *Journal of Financial Economics* **5**, 177-188.
- [28] Williams, D. (1991) *Probability with Martingales* (Cambridge: Cambridge University Press).