Notes on a 'Constructive Proof of the Existence of a Collateral Equilibrium'

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Abstract

In an interesting article in this Journal, Wei Ma (2015), claims that a constructive proof of existence of a Collateral Equilibrium (with a Leontief utility function) is provided in the paper. Moreover, there is also the additional claim that 'on the basis of this proof, we can (and we shall) develop an algorithm for computing that equilibrium' (*ibid*, p. 1). Thirdly, the statements that 'the algorithm is shown by simulation to be effective', and its effectiveness is demonstrated by 'the accuracy of the equilibrium it provided'. These claims are, from a recursion theoretic (or computability theory) point of view, meaningless.

Key words: Proof, Constructive, Effective, Existence

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The proof of existence of a Collateral Equilibrium (with a Leontief utility function), in Ma (2015) is *not* constructive.

The two theorems of relevance for this claim are Theorems 1 and 2, p. 23 (*ibid*).

The proof of the first theorem is given in the Appendix (pp. 23-26), and relies heavily (as the author candidly admits[1]) on the 'technique of proof' given in Dubey et. al. (2005).

This theorem is *non-constructive* for at least four mathematical reasons:

1. The appeal to Margaret Wright's non-constructive proof of Theorem 5 (p. 361, ff., in Wright, 1992, in turn, relies on the previous theorem, i.e., theorem 4, and its proof in *ibid*).
2. The appeal to Kakutani's fixed-point theorem (p.25), exactly along the lines of Dubey et. al. (*op.cit*)[2]. That the Kakutani fixed-point theorem is non-constructive was shown by one of us, in a recent contribution to this Journal (cf. Velupillai, 2015, p. 188).
3. The use of a non-constructive definition of a *compact set*, in the proof.
4. It is also possible to show that the mathematical logical reasoning used in the proof of Theorem 1 appeals to the *tertium non datur*, not just in finite cases. But this requires more careful dissection of the structure of the proof employed by Dubey et.al., on which Wei Ma crucially relies.

Theorem 2, apart from relying on the result that is Theorem 1, is more glaringly non-constructive in the following two senses:

i. It invokes an *undecidable disjunction* in the form of an appeal to the non-constructive Bolzano-Weierstrass theorem (p. 13, opening paragraph of the proof of Theorem 2).

[1] 'The technique of proof used here is analogous to that of Dubey et. al. (2005)'. Unfortunately the author also reproduces that pure existence proof – i.e., non-constructive (and non-effective or non-computable) – almost verbatim, in crucial places (cf. in particular, ibid, p. 25).

[2] But Dubey et.al. (op.cit) make no constructive claims for their proof, and for invoking the non-constructive – indeed, unconstructifiable – Kakutani fixed point theorem.

ii. In using the technique of *proof by contradiction*, hence invoking *the law of double-negation*, in non-finite cases (the second paragraph of the proof of Theorem 2, p. 13).

Finally, in characterizing an auxiliary economy (§ 5.1, pp. 14-15), 'to facilitate computation' – essentially to devise an algorithm, the author appeals to:

I. A 'construction' of 'smooth and positive functions' (first paragraph of § 5.1)[3].
II. The appeal to what the author refers to as the 'KKT conditions of each household' (ibid, p. 15), which are non-constructive.

These two reasons, at the least, invalidate the claim that the algorithm is effective (even in the narrow sense defined by the author). No such algorithm, relying on the proof of Theorems 1 and 2, can be shown to be 'effective by the accuracy of the equilibrium it provided'.[4]

We would like to end this brief note by expressing our adherence to Fred Richman's important observation (Richman, *op. cit*, p. 125; italics added):

> "Even those who like algorithms have remarkably little appreciation of the thoroughgoing algorithmic thinking that is required for *a constructive proof*. ...... I would guess that most realist mathematicians are unable even to recognize when a proof is constructive in the intuitionist's sense.
>
> *It is a lot harder than one might think to recognize when a theorem depends on a nonconstructive argument*. One reason is *that proofs are rarely self-contained*, but depend on other theorems whose proofs depend on still other theorems. These other theorems have often been internalized to such an extent that we are not aware whether or not nonconstructive arguments have been used, or must be used, in their proofs. Another reason is that the *law of excluded middle* [LEM] *is so ingrained in our thinking* that we do not distinguish between different formulations of a theorem that are trivially equivalent given LEM, although one formulation may have a constructive proof and the other not."

[3] Essentially by invoking, albeit implicitly, a non-constructive axiom of choice. See, also, the concluding quote from Fred Richman (1990), below.

[4] A careful reading of Scarf (1973, p.52) should warm any reader of the validity of such a statement. Moreover, the author's remarks on *comparative statics* (*ibid*, p. 2), are unwarranted, in the light of the non-constructive – and, uncomputable (constructive and computable are quite different mathematical concepts, especially from the point of view of devising algorithms for digital computers) – theorems and proofs in the paper. We would like to add that the author's remarks on comparative statics are analogous to, and as mathematically incorrect as, those by Shoven & Whalley (1992), on an exactly similar issue.

It is *not only* 'the law of the excluded middle is so ingrained in our thinking', but also the implicit appeals to non-constructive forms of the axiom of choice and the law of double negation – all of which are liberally used in Wei Ma's otherwise interesting article. It is just that its claims on constructive proofs, effective algorithms, and the connections between the former and the latter, are incorrect.

Conflict of Interest: The authors declare that they have no conflict of interest.

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